SYMBOLIC POWERS IN ALGEBRA AND GEOMETRY

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Abstract. These are notes for a lecture series given at Şcoala Nationala de Algebra held at the University of Bucharest in June 2023 . All misprints belong to the author. In an attempt to limit them, please let me know if you find any errors or typos, however small.

Assumptions

Throughout, all rings are commutative with identity. Moreover, we will almost always assume our rings are noetherian, so we settle on the convention that rings are noetherian unless we say otherwise.

We will assume the reader has some knowledge of elementary commutative algebra, as in [AM69], [Mat80], [Mat89], [BH93], or [Eis95].

$\mathbb{N}$ denotes the set of natural numbers, including zero.

1. Ordinary and Symbolic Powers of Ideals

Set up 1.1. Throughout R is a commutative noetherian ring and I is an ideal of R.

1.1. Ordinary Powers.

Definition 1.2. The **(ordinary) powers** of I, one for each $n \in \mathbb{N}$, are the ideals generated by n-fold products of (not necessarily distinct) elements of I

$$I^n = (f_1 \cdot f_2 \cdots f_n : f_i \in I)$$

It is worth emphasizing I^n is merely generated by products of n elements of I, but the that a typical element of I^n is *not* a sum of such products.

It is easy to find a finite set of (not necessarily minimal) generators of I.

Remark 1.3. If $I = (f_1, \ldots, f_t)$ then for each $n \in \mathbb{N}$ we have

$$I^n = (f_{i_1} \cdot f_{i_2} \cdots f_{i_n} : 1 \leq i_1 \leq i_2 \leq \ldots \leq i_n \leq t).$$

Properties 1.4 (Properties of ordinary powers)**.**

(1) $I^0 = R$, $I^1 = I$
(2) $I^b \subseteq I^a \iff a \leq b$
(3) $I^a \cdot I^b = I^{a+b}$.

To explain the downside of the ordinary powers it is worth taking a detour into the theory of associated prime ideals.

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Definition 1.5. The **associated primes** of I (this is well-established terminology abuse because this really refers to the associated primes of R/I) are the ideals $\mathrm{Ann}_R(x)$ where $x \in R/I$. The **set of associated primes** of I is denoted $\mathrm{Ass}(I)$.

Definition 1.6. Special among the associated primes are the **minimal primes** of I, which are prime ideals P such that $I \subseteq P$ and whenever P' is a prime ideal such that $I \subseteq P' \subseteq P$ then $P' = P$. Equivalently the minimal primes of I are the minimal elements of $\mathrm{Ass}(I)$ with respect to containment. The **set of minimal primes** of I is denoted $\mathrm{Min}(I)$.

Remark 1.7 (Geometric interpretation of the minimal primes)**.** The identity

$$\sqrt{I} = \bigcap_{P \in \mathrm{Min}(I)} P$$

signifies the decomposition of $V(I)$ into irreducible components

$$V(I) = \bigcap_{P \in \mathrm{Min}(I)} V(P).$$

Taking powers preserves the minimal primes.

Exercise 1. Prove that for each $n \in \mathbb{N}$, $\mathrm{Min}(I^n) = \mathrm{Min}(I)$.

In sharp contrast to Exercise 1, the associated primes are more unruly when taking ordinary powers.

Example 1.8. Let $R = K[x, y, z]$ and $I = (xy, xz, yz) = (x, y) \cap (x, z) \cap (y, z)$. Then

$$I^2 = (x, y)^2 \cap (x, z)^2 \cap (y, z)^2 \cap (x^2, y^2, z^2),$$

so

$$Ass(I) = \{(x, y), (x, z), (y, z)\}$$

and

$$\mathrm{Ass}(I^2) = \{(x, y), (x, z), (y, z), (x, y, z)\}.$$

This presents a problem form a geometric viewpoint: while $V(I)$ in this example is the union of the three coordinate lines in K^3 if we think affinely (or the three coordinate points in $\mathbb{P}^3$ if we think projectively), $V(I^2)$ picks up an additional embedded point at the origin (or a geometrically irrelevant component when thinking projectively).

The symbolic powers we introduce next are meant to deal with this inconsistency.

1.2. **Symbolic Powers.** There are the definition of symbolic powers in the literature.

Definition 1.9. The **(symbolic) powers** of I, one for each $n \in \mathbb{N}$, are either one of the following two families of ideals

$$I^{(n)}_{\mathrm{Min}} = \bigcap_{P \in \mathrm{Min}(I)} (I^n R_P \cap R) = \{f \in R : sr \in I^n \text{ for some } s \notin \bigcup_{P \in \mathrm{Min}(I)} P\}.$$

or

$$I^{(n)}_{\mathrm{Ass}} = \bigcap_{P \in \mathrm{Ass}(I)} (I^n R_P \cap R) = \{f \in R : sr \in I^n \text{ for some } s \notin \bigcup_{P \in \mathrm{Ass}(I)} P\}.$$

Why are there two different definitions? Each version satisfies a desirable property (see Properties 1.10 item 1 and item 8 below) so they are both in some sense natural. The two definitions agree whenever I is has no embedded primes, which is the geometrically relevant case; see Properties 1.10 item 4,

Whenever I leave out the subscript Min or Ass I mean that either version satisfies a given statement.

Properties 1.10 (Properties of ordinary powers)**.**

(1) $I^0 = R$
(2) $I^{(1)}_{\mathrm{Ass}} = I$, $I^{(1)}_{\mathrm{Min}} = I$ if and only if $\mathrm{Ass}(I) = \mathrm{Min}(I)$
(3) $I^n \subseteq I^{(n)}$ for all $n \in \mathbb{N}$
(4) $I^{(n)}_{\mathrm{Ass}} = I^{(n)}_{\mathrm{Min}}$ for all $n \in \mathbb{N}$ if and only if $\mathrm{Ass}(I) = \mathrm{Min}(I)$
(5) $I^{(b)} \subseteq I^{(a)} \iff a \leq b$
(6) $I^{(a)} \cdot I^{(b)} = I^{(a+b)}$
(7) $\mathrm{Min}(I^{(n)}) = \mathrm{Min}(I)$
(8) $\mathrm{Ass}(I^{(n)}_{\mathrm{Min}}) = \mathrm{Min}(I)$
(9) $\mathrm{Ass}(I^{(n)}_{\mathrm{Ass}}) \supseteq \mathrm{Ass}(I)$ with equality if $\mathrm{Ass}(I) = \mathrm{Min}(I)$.

In contrast to Remark 1.3 it is difficult to find generators for the symbolic powers of an ideal. Nevertheless here are some ideas that help.

Properties 1.11 (Computing symbolic powers)**.**

(1) if I is generated by a regular sequence then $I^{(n)} = I^n$
(2) if $I = P_1 \cap \cdots \cap P_r$ with P_i distinct prime ideals, then

$$I^{(n)} = P_1^{(n)} \cap \cdots \cap P_r^{(n)}.$$

(3) $I^{(n)}_{\mathrm{Min}}$ is obtained from I^n by removing primary components of primes *not* in $\mathrm{Min}(I)$
(4) $I^{(n)}_{\mathrm{Min}}$ is obtained from I^n by removing primary components of primes that are not contained in some $P \in \mathrm{Ass}(I)$.

Combining the above bullet points gives a convenient way to compute the symbolic powers for monomial ideals and ideals defining points (more generally unions of linear subspaces) in affine or projective space.

Remark 1.12. Suppose I is either a square-free momomial ideal or an ideal defining points in K^d or $\mathbb{P}^d$. Then $I = P_1 \cap \cdots \cap P_r$ with P_i distinct prime ideals, each generated by a regular sequence and thus

$$I^{(n)} = P_1^n \cap \cdots \cap P_r^n, \text{ for all } n \in \mathbb{N}.$$

Example 1.13. Let $R = K[x, y, z]$ and $I = (xy, xz, yz) = (x, y) \cap (x, z) \cap (y, z)$. Then

$$I^{(2)} = (x, y)^2 \cap (x, z)^2 \cap (y, z)^2.$$

This can be seen by combining Properties 1.11 item 1 and item 2 or by utilizing Example 1.8 and Properties 1.11 item 4 or item 5.

Remark 1.14. Another class of ideals for which the symbolic powers are well understood is that of determinantal ideals. Explicit descriptions of the symbolic and ordinary powers of generic determinantal ideals are given in [DEP80]. In fact, such a description also exists for ideals of minors of generic symmetric matrices and for pfaffians of skew-symmetric matrices

[JMnV15, DN96]. Any such ideal and all of its symbolic powers are generated by minors of X of various sizes (not just t). Moreover, there is a basis for $k[X]$ given by certain products of minors of X, called *standard monomials*, and thus one can explicitly describe $I^{(n)}$ for all n by determining which standard monomials live in $I^{(n)}$ [DEP80, JMnV15, DN96].

1.3. **Differential Powers and the Zariski-Nagata Theorem.** The geometric meaning of the symbolic powers is a strong reason to study them. The Nulstellensatz tells us that if $I(=\sqrt{I})$ is a radical ideal in a polynomial ring $R = K[x_1, \ldots, x_n]$ then

$$I = \bigcap_{\substack{\mathfrak{m} \supseteq I \\ \mathfrak{m} \text{ maximal ideal of } R}} \mathfrak{m}. \tag{1.1}$$

The Zariski–Nagata theorem is a higher order version of this result, which says that the symbolic powers of a radical ideal are the sets of polynomials that vanish up to order n on the corresponding variety. There are actually a few different results known as Zariski–Nagata; the first one is a theorem of Nagata's [Nag62].

Theorem 1.15 (Zariski–Nagata [Zar49], Eisenbud–Hochster [EH79])**.** *Let K be a perfect field and $R = K[x_1, \ldots, x_d]$. For any radical ideal I, we have*

$$I^{(n)} = \bigcap_{\substack{\mathfrak{m} \supseteq I \\ \mathfrak{m} \text{ maximal ideal of } R}} \mathfrak{m}^n.$$

Here one should think of the elements of the ideal $\mathfrak{m}^n$ as polynomial functions vanishing to order n at the point $V(\mathfrak{m}) \in V(I)$.

To prove the Zariski-Nagata theorem, we will proceed by double containment. One containment requires some facts about Hilbert-Samuel multiplicity. The Hilbert-Samuel multiplicity is an important invariant which detects and measures singularities.

Definition 1.16. Let $(R, \mathfrak{m})$ be a local ring of dimension d. Let $\lambda(M)$ denote the length of the module M. The **Hilbert-Samuel multiplicity** of R is

$$e(R) = \lim_{n \to \infty} \frac{d!\lambda(R/\mathfrak{m}^n)}{n^d}.$$

The defining limit exists, and can also be described in terms of the Hilbert function of $\operatorname{gr}_{\mathfrak{m}} M$: the Hilbert function is eventually a polynomial, and $e(R)$ is $d!$ times the coefficient of the highest order term in that polynomial. We will need a few facts about $e(R)$, which we will not prove for now:

- If $(R, \mathfrak{m})$ is a regular local ring and $f \in \mathfrak{m}$, then $e(R) = \operatorname{ord}(f) = \max\{t \mid f \in \mathfrak{m}^t\}$.
- Under mild assumptions, $e(R) \geqslant e(R_P)$.

Using these two facts, we can now prove Nagata's version [Nag62] of Theorem 1.15.

Lemma 1.17 (Local Zariski–Nagata)**.** *Let $(R, \mathfrak{m})$ be a regular local ring. For every prime ideal P and every $n \geqslant 1$,*

$$P^{(n)} \subseteq \mathfrak{m}^n.$$

Proof. Fix a prime ideal P and an element $f \in \mathfrak{m}$. First, note that R_P is also regular, and that $f \in P^{(t)}$ if and only if $\frac{f}{1} \in P^t R_P$, so by the properties above,

$$\max\{t \mid f \in P^{(t)}\} = \max\{t \mid \frac{f}{1} \in P^t R_P\} = e((R/f)_P) \leqslant e(R/f) = \max\{t \mid f \in \mathfrak{m}^t\}.$$

So if $f \in P^{(n)}$, then we must have $f \in \mathfrak{m}^n$, showing that $P^{(n)} \subseteq \mathfrak{m}^n$. ∎

Definition 1.18. Given a finitely generated k-algebra R, the **k-linear differential operators** on R of order n, $D_R^n \subseteq \operatorname{Hom}_k(R,R)$, are defined as follows:

- The differential operators of order zero are the k-linear maps which are also R-linear:
$$D^0_{R|k} = \operatorname{Hom}_R(R,R) \cong R.$$
- We say that $\delta \in \operatorname{Hom}_k(R,R)$ is an operator of order up to n, meaning $\delta \in D_R^n$, if
$$[\delta, r] = \delta r - r\delta$$
is an operator of order up to $n-1$ for all $r \in D_R^0$.

The **ring of k-linear differential operators** is the subring of $\operatorname{Hom}_k(R,R)$ defined by
$$D_{R|k} = \bigcup_{n \in \mathbb{N}} D^n_{R|k}.$$
In particular, the multiplication on $D_{R|k}$ just composition.

If R or k are clear from the context, we may drop one of the subscripts, or both. Notice that $D_{R|k}$ is almost always a noncommutative ring!

Example 1.19. Let K be a field and $R = K[x_1, \ldots, x_d]$ or $R = K[[x_1, \ldots, x_d]]$. When K is a field of characteristic 0,
$$D_R^n = \bigoplus_{\alpha_1 + \ldots + \alpha_d \leq n} R \cdot \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} \quad \text{and} \quad D_{R|K} = R\left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d} \right\rangle.$$
When K has prime characteristic p, things are a little more complicated; notice that over $R = K[x]$, $\frac{\partial^p}{\partial x^p}(x^n) = 0$ for any n, but there are indeed nonzero differential operators of order p. To give a correct description of our differential operators on $R = K[x_1, \ldots, x_d]$ over any field k of any characteristic, we consider
$$D_\alpha = \frac{1}{\alpha_1!}\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{1}{\alpha_d!}\frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}, \qquad \text{where} \quad D_\alpha(x^\beta) = \begin{cases} \binom{\beta}{\alpha} x^{\beta-\alpha} & \text{if } \alpha_i \geqslant \beta_i \text{ for all } i \\ 0 & \text{otherwise.} \end{cases}$$
and now we have
$$D_R^n = \bigoplus_{\alpha_1 + \cdots + \alpha_d \leq n} D_\alpha.$$
For example, when $R = \mathbb{F}_3[x]$, $D_3(x^5) = \binom{5}{3}x^{5-3}$; since $\binom{5}{3} = \frac{5!}{3!2!} = \frac{5 \cdot 4}{2} = 10$, and $10 = 1$ in $\mathbb{F}_3$, this means that $D_3(x^5) = x^2$. On the other hand, notice that $\frac{\partial^3}{\partial x^3}(x^5) = 5 \cdot 4 \cdot 3 \cdot x^2 = 0$.

Definition 1.20. Let R be a finitely generated K-algebra, I an ideal of R, and n be a positive integer. The n^{th} K-linear **differential power** of I is given by
$$I^{\langle n \rangle} = \{ f \in R \,|\, \delta(f) \in I \text{ for all } \delta \in D_R^{n-1} \}.$$

Exercise 2. Let R be a finitely generated k-algebra, I be an ideal of R, and n be a positive integer. The set $I^{\langle n \rangle}$ is an ideal.

Exercise 3. Let $\{I_\alpha\}_{\alpha \in A}$ be an indexed family of ideals. For every $n \geqslant 0$ we have
$$\bigcap_{\alpha \in A} I_\alpha^{\langle n \rangle} = \left(\bigcap_{\alpha \in A} I_\alpha \right)^{\langle n \rangle}.$$

Properties 1.21. Let R be a finitely generated K-algebra, I be an ideal of R.

(1) For all $n \in \mathbb{N}$ the set $I^{\langle n \rangle}$ is an ideal.
(2) $I^n \subseteq I^{\langle n \rangle}$ for all $n \in \mathbb{N}$
(3) $I^{\langle b \rangle} \subseteq I^{\langle a \rangle} \iff a \leq b$
(4) $I^{\langle a \rangle} \cdot I^{\langle b \rangle} \subseteq I^{\langle a+b \rangle}$
(5) If P is a prime ideal, then $P^{\langle n \rangle}$ is P-primary for all $n \geq 1$.
(6) For any radical ideal I and prime ideal P, $(I_P)^{\langle n \rangle} = \left(I^{\langle n \rangle}\right)_P$.

An important connection between the symbolic and differential powers is that they agree for radical ideals in polynomial and power series rings.

Theorem 1.22. *Let K be a perfect field, $R = K[x_1, \dots, x_d]$ or $R = K[[x_1, \dots, x_d]]$. Then for every $n \in \mathbb{N}$ the following hold*

(1) if $\mathfrak{m}$ is a maximal ideal, then $\mathfrak{m}^{\langle n \rangle} = \mathfrak{m}^n$
(2) if P is prime ideal, then $P^{(n)} = P^{\langle n \rangle}$
(3) if I is a radical ideal, then $I^{(n)} = I^{\langle n \rangle}$

Proof. (1) If $f \notin \mathfrak{m}^n$, then f has a monomial of the form $\mu = x_1^{a_1} \cdots x_d^{a_d}$, with nonzero coefficient $c \in K$, which is minimal among all monomials appearing in f under the graded lexicographical order. Then the differential operator

$$\Delta = \frac{1}{a_1!}\frac{\partial}{\partial x_1} \cdots \frac{1}{a_d!}\frac{\partial}{\partial x_d}$$

maps the term $cx_1^{a_1} \cdots x_d^{a_d}$ to c and all other terms appearing in f either to a non constant monomial or to zero. Consequently,

$$\Delta(f) = c + \text{ terms in } \mathfrak{m} \qquad \text{shows} \qquad \Delta(f) \notin \mathfrak{m}$$

and thus $f \notin \mathfrak{m}^{\langle n \rangle}$. Hence we have obtained (the contrapositive of) $\mathfrak{m}^{\langle n \rangle} \subseteq \mathfrak{m}^n$. The other containment follows from Properties 1.21 (2).

□

We are now ready to give a proof of Theorem 1.15 in the case when K is perfect.

Proof of Theorem 1.15. We can write I as the intersection of finitely many primes, say

$$I = P_1 \cap \cdots \cap P_r.$$

On the one hand,

$$I^{(n)} = P_1^{(n)} \cap \cdots \cap P_r^{(n)} \subseteq \bigcap_{\substack{\mathfrak{m} \supseteq I \\ \mathfrak{m} \text{ maximal ideal of } R}} \mathfrak{m}^n$$

is true since the containment $P_i^{(n)} \subseteq \mathfrak{m}^n$ is true for every P_i and every $\mathfrak{m}$ appearing in the intersection on the right after localizing at $\mathfrak{m}$ by Lemma 1.17 .

For the converse, take $f \in \mathfrak{m}^n$ for all the maximal ideals $\mathfrak{m} \supseteq I$. For each maximal ideal $\mathfrak{m}$ containing I, we have $f \in \mathfrak{m}^{\langle n \rangle}$ by Theorem 1.22 (1), so for every $\partial \in D^{n-1}$, $\partial(f) \in \mathfrak{m}$. By (1.1) we have

$$I = \bigcap_{\substack{\mathfrak{m} \supseteq I \\ \mathfrak{m} \in \text{ maximal ideal}}} \mathfrak{m},$$

thus $\partial(f) \in I$ for every $\partial \in D^{n-1}$, so $f \in I^{\langle n \rangle}$. By Theorem 1.22, $I^{\langle n \rangle} = I^{(n)}$, so it follows that

$$\bigcap_{\substack{\mathfrak{m} \supseteq I \\ \mathfrak{m} \text{ maximal ideal of } R}} \mathfrak{m}^n \subseteq I^{(n)}.$$

□

While our proof of Theorem 1.15 really requires that k be a perfect field, since it uses the differential operators version of Zariski-Nagata, we note that this holds for any general field, as shown by Eisenbud and Hochster [EH79]. A proof of the general case can also be found in [DDSG+17, Theorem 2.12].

2. The containment problem

2.1. A uniform constant.

The containment problem for ordinary and symbolic powers of ideals is the following

Problem 2.1. *(Containment Problem). Let I be an ideal in a noetherian ring R. For what values of a and b is $I^{(a)} \subseteq I^b$?*

To solve such a problem it helps to use the following local criterion for containments

Exercise 4. (Containments are local) Given ideals I and J in a ring R, $I \subseteq J$ if and only if $I_P \subseteq J_P$ for all primes $P \in \operatorname{Ass}(R/J)$.

This question first appeared in work of Schenzel in the 1980s [Sch86], but gained more serious attention in the new millennium, in large part due to Irena Swanson's answers [Swa00] to some of Schenzel's questions, which then inspired work of Ein–Lazarsfeld–Smith and Hochster–Huneke [ELS01, HH02] in the early 2000s.

Theorem 2.2 (Swanson, 2000 [Swa00])**.** *Let R be a noetherian local ring and I an ideal in R. The following are equivalent:*

(1) For every $b \in \mathbb{N}$ there is an $a \in \mathbb{N}$ such that $I^{(a)} \subseteq I^b$?
(2) There exists a constant s such that $I^{(sn)} \subseteq I^n$ for all n.

Swanson's proof left open the problem of finding a formula for s. Soon after Swanson's theorem was announced, Lawrence Ein, Robert Lazarsfeld, and Karen Smith [ELS01] found that Swanson's constant can be computed very explicitly when $R = \mathbb{C}[x_1, \ldots, x_d]$, or more generally when R is any smooth $\mathbb{C}$-algebra. Their proof relies on multiplier ideals and their wonderful properties. Proving beautiful theorems is contagious, it appears, since soon after that Mel Hochster and Craig Huneke [HH02] extended the result to any regular ring containing a field, using completely different techniques: they proved the result in prime characteristic, and then extended it to equicharacteristic zero via reduction to characteristic p. Later Linquan Ma and Karl Schwede [MS18] complete the story by proving the result holds in mixed characteristic. The key idea is to establish a mixed characteristic version of multiplier ideals, which then allows one to follow Ein, Lazersfeld, and Smith's original strategy. One technical detail remained to be settled: Ma and Schwede's proof needed R to be excellent. But recently Takumi Murayama [Mur21] settled the final technical details that allows us to state the following beautiful theorem:

Definition 2.3. The **big height** of an ideal I is the maximum height of an associated prime of I.

Theorem 2.4 (Ein–Lazarsfeld–Smith, Hochster–Huneke, Ma–Schwede, Murayama)**.** *Let R be a regular ring, and I be a radical ideal. If I has big height h, then*

$$I^{(hn)} \subseteq I^n \qquad \text{for all } n \geqslant 1.$$

In this section, we will discuss a key ingredient for Hochster and Huneke's proof of Theorem 2.4 in prime characteristic. One of the main advantages of characteristic p is that while we cannot control the associated primes of the powers of an ideal, when the ring is regular we can control the associated primes of the Frobenius powers.

Definition 2.5. The **Frobenius powers** of an ideal I, one for each $n \in \mathbb{N}$, are the sets of n-th powers of elements of I

$$I^{[n]} = \{f^n : f \in I\}.$$

Lemma 2.6. *Let R be a regular ring of prime characteristic p and I be an ideal in R. Then* $\operatorname{Ass}\left(I^{[q]}\right) = \operatorname{Ass}(I)$ *for all $q = p^e$.*

The following theorem is the key piece of the puzzle, even though it is beautifully elementary: it is just a fancy version of the Pigeonhole Principle.

Theorem 2.7. *Let I be a radical ideal of big height h in a regular ring R of prime characteristic p. For all $q = p^e$, $I^{(hq)} \subseteq I^{[q]}$.*

Proof. Fix $q = p^e$. By Exercise 4, it is sufficient to show the containment holds after localizing at all the associated primes of $I^{[q]}$ which are given by $\operatorname{Ass}\left(I^{[q]}\right) = \operatorname{Ass}(I)$ according to Lemma 2.6.

Let $Q \in \operatorname{Ass}(I)$. We know that $I_Q = Q_Q$, $\left(I^{[q]}\right)_Q = Q_Q^{[q]}$, and $\left(I^{(hq)}\right)_Q = Q_Q^{hq}$. By assumption, $\dim(R_Q) = \operatorname{height}(Q) \leqslant h$. So we have a regular local ring of dimension h with maximal ideal $\mathfrak{m}$ and we want to show that $\mathfrak{m}^{hq} \subseteq \mathfrak{m}^{[q]}$. Since our ring is regular, $\mathfrak{m}$ is generated by dimension many elements, so at most h elements. Let $\mathfrak{m} = (x_1, \ldots, x_h)$. The power $\mathfrak{m}^{hq}$ is generated by all monomials of the form $x_1^{a_1} \cdots x_h^{a_h}$ with

$$a_1 + \cdots + a_h \geqslant hq,$$

so by the Pigeonhole Principle, there exists i such that $a_i \geqslant q$. Therefore,

$$x_1^{a_1} \cdots x_h^{a_h} \in (x_i^q) \subseteq \mathfrak{m}^{[q]}. \quad \square$$

Remark 2.8. While the statement of Theorem 2.7 is all we need to prove Theorem 2.4, we note that if we apply the full power of the Pigeonhole Principle we can do even better: indeed, as long as

$$a_1 + \cdots + a_h \geqslant h(q-1) + 1,$$

at least one $a_i \geqslant q$. We conclude that any radical ideal of big height h satisfies $I^{(hq-h+1)} \subseteq I^{[q]}$ for all $q = p^e$.

We do not present here the full proof of Hochster and Huneke of Theorem 2.4. To show the theorem in positive characteristic they use Theorem 2.7 and the theory of tight closure. To show the theorem in equicharacteristic zero, one uses standard reduction to characteristic p techniques — which are quite technical, and thus we will not discuss them here.

The following generalization is [ELS01, Theorem 2.2] in the case of smooth complex varieties, [HH02, Theorem 2.6] in equicharacteristic, and [Mur21] in mixed characteristic:

Theorem 2.9 (Ein–Lazersfeld–Smith, Hochster–Huneke, Murayama)**.** *Let I be a radical ideal of a regular ring, and let h be the big height of I. Then for all $n \geqslant 1$ and all $k \geqslant 0$,*

$$I^{(hn+kn)} \subseteq \left(I^{(k+1)}\right)^n.$$

One can even do better! If we reinterpret kn as a sum of n terms all equal to k, we can extend this further and allow the sum of any n terms:

Theorem 2.10 (Johnson, Murayama [Joh14, Mur21])**.** *Let I be a radical ideal of a regular ring, and let h be the big height of I. Then for all $n \geqslant 1$ and all $a_1, \ldots, a_n \geqslant 0$,*

$$I^{(hn+a_1+\cdots+a_n)} \subseteq I^{(a_1+1)} \cdots I^{(a_n+1)}.$$

2.2. **Harbourne's conjecture.** Theorem 2.4 says that if I is any radical ideal of big height h, then $I^{(hn)} \subseteq I^n$ for all $n \geqslant 1$. But does this completely answer the Containment Problem Problem 2.1? A complete answer would find for any given $b \in \mathbb{N}$ the smallest a such that $I^{(a)} \subseteq I^b$, and the theorem tells us only that $a \leqslant hb$. But how much smaller than hb can a be?

The case $b = 1$ is easily solved: one can take $a = 1$. The case $b = 2$ is already challenging.

Example 2.11. Consider the monomial ideal $I = (xy, xz, yz)$ in $R = k[x, y, z]$, where k is a field. Since all the minimal primes of I have height 2, the big height of I is 2. We saw in Example 1.13 that $I^{(2)} \neq I^2$, and Theorem 2.4 says that $I^{(2n)} \subseteq I^n$ for all n. To solve the containment problem for I^2 we need only to determine whether $I^{(3)} \subseteq I^2$. And indeed, one can easily show that this does hold. Thus we have completely solved the containment problem for this ideal and for $b = 2$: $I^{(a)} \subseteq I^2$ if and only if $a \geq 3$.

Craig Huneke asked the question of whether one can always do better than predicted by Theorem 2.9 when $h = b = 2$.

Question 2.12 (Huneke, 2000)**.** Let P be a prime ideal of height 2 in a regular local ring R. Is $P^{(3)} \subseteq P^2$?

This question remains open. The symbolic powers of the primes determining curves of the form (t^a, t^b, t^c) exhibit lots of interesting behavior, and it turns out they do satisfy the result.

Theorem 2.13 (Grifo, 2020 [Gri20])**.** *Let k be a field of characteristic not 3. If P is the prime ideal in $k[x, y, z]$ defining the curve (t^a, t^b, t^c), then $P^{(3)} \subseteq P^2$.*

Brian Harbourne extended Huneke's question to a much more general setting. We note that his original question, which first appeared in print in [BDRH+09, 8.4.3], was about homogeneous ideals in $R = k[x_1, \ldots, x_d]$, though we present here a slightly modified version of his question:

Conjecture 2.14 (Harbourne, 2008)**.** *Let I be a radical ideal in a regular ring R. If I has big height h, then*

$$I^{(hn-h+1)} \subseteq I^n \qquad \text{for all } n \geqslant 1.$$

The value suggested by this conjecture is very natural. In fact, Remark 2.8 can be used to show that if R has characteristic p and we take $n = q = p^e$ for some e, then the containment in Habourne's Conjecture holds, and it is simply the value suggested by the Pigeonhole Principle. The same argument also works for monomial ideals.

Exercise 5. Use the pigeonhole principle to show that if I is an ideal generated by monomials in a polynomial ring then Harbourne's Conjecture 2.14 holds.

Conjecture 2.14 holds for finite sets of generic points in the plane.

Definition 2.15. A set S of points in $\mathbb{P}^n_k$ is **generic** if the coordinates of the points in S are algebraically independent over the prime field of k; this is the smallest subring of k containing 1, which is either isomorphic to $\mathbb{F}_p$ or $\mathbb{Q}$ depending on the characteristic of k.

Theorem 2.16 (Bocci–Harbourne, 2010, Theorems 4.1 and 4.2 in [BH10a])**.** *Let S be a set of generic points in $\mathbb{P}^2$, and let $I = I(S)$. Then $I^{(3)} \subseteq I^2$. Moreover, $I^{(a)} \subseteq I^b$ whenever $\frac{a}{b} > \frac{3}{2}$.*

Notice that $2n - 1 > \frac{3}{2}n$ for all $n \geqslant 2$, so in particular generic points in $\mathbb{P}^2$ satisfy Harbourne's Conjecture 2.14. This also holds in projective 3-space.

Theorem 2.17 (Dumnicki, 2015 [Dum15])**.** *Let S be a set of generic points in $\mathbb{P}^3$, and let $I = I(S)$. Then I satisfies Harbourne's Conjecture: $I^{(3n-2)} \subseteq I^n$ for all $n \geqslant 1$.*

There are examples that show that in general for any given $n \in \mathbb{N}$ we cannot do better than Harbourne's Conjecture's predicts for all ideals.

Exercise 6. Let $R = k[x_1, \ldots, x_d]$ and consider the squarefree monomial ideal

$$I = \bigcap_{i<j} (x_i, x_j)\,.$$

Show that while $I^{(2n-1)} \subseteq I^n$ holds for all $n \geqslant 1$, $I^{(2n-2)} \nsubseteq I^n$ for $n < d$, so we cannot do better than Harbourne's Conjecture in this case. What happens when $n = d$? How does this example generalize to higher height?

In contrast with Theorem 2.16 however it turns out that Harbourne's Conjecture 2.14 does not hold for all radical ideals — not even for all ideals that define finite sets of points in the plane.

The first counterexample was found by Dumnicki, Szemberg, and Tutaj-Gasińska [DSTG13], and it is the radical ideal defining a certain nice configuration of twelve points in $\mathbb{P}^2$ over $\mathbb{C}$. Harbourne and Seceleanu extended their example to a family of examples in any characteristic other than 2 [HS15].

Example 2.18 (Dumnicki—Szemberg—Tutaj-Gasińska, 2013, Harbourne–Seceleanu, 2015 [DSTG13, HS15])**.** Fix $n \geqslant 3$. Let K be a field of characteristic not 2 and containing n distinct roots of unity, and let $R = K[x, y, z]$. The ideal

$$I = (x(y^n - z^n), y(z^n - x^n), z(z^n - x^n))$$

is a radical ideal of height 2, and yet $I^{(3)} \nsubseteq I^2$. In fact, the element

$$f = (y^n - z^n)(z^n - x^n)(z^n - x^n)$$

satisfies $f \in I^{(3)}$, but $f \notin I^2$.

This ideal is the homogeneous radical ideal corresponding to a particularly nice configuration of points in $\mathbb{P}^2$, know as the **Fermat configuration**. When $n = 3$, the corresponding picture shown in Figure 1.

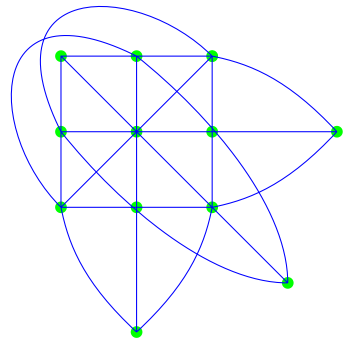

FIGURE 1. Fermat configuration of points when $n = 3$.

Other counterexamples to Harbourne's Conjecture have since been found. There is much we do not understand about the conjecture, though many of the known counterexamples arise as the singular loci of hyperplane arrangements, and Drabkin and Seceleanu [DS20] have completely classified which finite complex reflection group lead to counterexamples to $I^{(3)} \subseteq I^2$.

Notice, however, that all of the above mentioned examples do not provide any counterexamples to Huneke's Question 2.12. Nor do they provide counterexamples to $I^{(hn-h+1)} \subseteq I^n$ for any $n > 2$. In view of this it is natural to ask whether Harbourne's conjecture may still be valid for $n \gg 0$. This expectation has been formalized as a conjecture by Grifo.

Conjecture 2.19 (Stable Harbourne conjecture by Grifo [Gri20])**.** *Let I be a radical ideal of big height h in a regular ring R. Then $I^{(hn-h+1)} \subseteq I^n$ for all n sufficiently large.*

Exercise 7. Prove that Conjecture 2.19 holds
(1) if $I^{(hm-h)} \subseteq I^m$ holds for some value m (this was proven in [Gri20] using Theorem 2.10),
(2) if $I^{(hm-h+1)} \subseteq I^m$ holds for some m and $I^{(n+h)} \subseteq II^{(n)}$ holds for all $n > m$,
(3) if $\rho(I) < h$ (see Definition 3.22).

If we allow h to grow with n then there exist positive characteristic examples for which the containments in Conjecture 2.14 fail for arbitrarily large values of n. Note that R, I, and h change as n grows however. So this is not a counterexample to Conjecture 2.19.

Example 2.20 (Harbourne–Seceleanu, 2015 [HS15])**.** Let k be a finite field of characteristic $p > 0$ and let $K \supset k$ be any field. Let X denote the set of all points of $\mathbb{P}^N_K$ having all coordinates in k, except for any one of these points. Then $h = N$ and $I^{(Nn-N+1)} \not\subseteq I^n$ whenever

- $p > 2, n = 2$ and $N = (p+1)/2$ or
- $n = (p+N-1)/N$ in which case $Nn - N + 1 = p$, $p > (N-1)^2$ and $p \equiv 1 \pmod{N}$.

In positive characteristic, however, Conjecture 2.14 does hold for special classes of ideals which exhibit particularly nice properties of the Frobenius map.

Definition 2.21 (Hochster—Roberts, 1974 [HR74])**.** Let R be a ring of prime characteristic p. We say R if **F-pure** if the Frobenius map is pure, meaning that for every R-module M, setting $F_*(M)$ to be M with R-action $rm = r^p m$, the identity map $M \to F_*(M)$ is injective.

Remark 2.22. A stronger property than F-pure is that the Frobenius map $F : R \to R$ splits.

Example 2.23. Examples of F-pure rings include

- quotients of polynomial rings by square-free monomial ideals
- quotients of polynomial rings by generic determinantal ideals
- Veronese subrings of polynomial rings
- nice rings of invariants of linearly reductive groups

The F-purity condition is a measure of *nice* singularities, and yet the class of F-pure rings is quite large, containing all Stanley-Reisner rings and all direct summands of F-finite regular rings. In an R-pure ring, the Frobenius map has *one* splitting; if it has *many* splittings, our rings has very nice singularities.

Definition 2.24 (Hochster–Huneke [HH89])**.** Let R be a reduced F-finite ring (meaning that the Frobenius map is module-finite) of prime characteristic p. We say R is **strongly F-regular** if for every $c \in R$ that is not in any minimal prime of R, there exists $e \gg 0$ such that the map $R \to R^{1/p}$ sending $1 \mapsto c^{1/p}$ splits.

Theorem 2.25 (Grifo–Huneke, 2019 [GH19])**.** *Let R be a regular ring of prime characteristic p and dimension d. Let I be an ideal in R of big height h.*

(1) If R/I is F-pure, then for all $n \geqslant 1$ we have $I^{(n+h)} \subseteq II^{(n)}$.
In particular, $I^{(hn-h+1)} \subseteq I^n$ for all $n \geq 1$.
(2) If $h \geqslant 2$ and R/I is strongly F-regular, then $I^{(d)} \subseteq II^{(d+1-h)}$ for all $d \geqslant h-1$.
In particular, $I^{((h-1)n-(h-1)+1)} \subseteq I^n$ for all $n \geq 1$.

For primes of height 2, Theorem 2.25 (2) actually gives equality:

Corollary 2.26. *Let R be a regular ring of characteristic $p > 0$, and I a height 2 prime such that R/I is strongly F-regular. Then $I^{(n)} = I^n$ for all $n \geqslant 1$.*

This gives non-trivial classes of ideals with $I^{(n)} = I^n$ for all $n \geqslant 1$.

Example 2.27. Let $S = k[s^3, s^2t, st^2, t^3] \subseteq k[s,t]$, where k is a field of characteristic $p > 3$. This is a Veronese subring of $k[s,t]$, and thus strongly F-regular. We can write S as a quotient of $k[a,b,c,d]$ by a 3-generated height 2 prime ideal,

$$P = \left(b^2 - ac, c^2 - bd, bc - ad\right).$$

By Corollary 2.26, $P^{(n)} = P^n$ for all $n \geqslant 1$.

3. Asymptotic Invariants

3.1. Interpolation Problems.

Set up 3.1. In this section $R = \bigoplus_{i \in \mathbb{N}} R_i$ will be a graded ring and I will be a graded ideal of R. The ideal $\mathfrak{m} = \bigoplus_{i \geq 1} R_i$ is the graded maximal ideal of R.

In this section we return to the geometric roots of the symbolic powers. Recall that the Zariski–Nagata Theorem 1.15 says that the n-th symbolic power of a radical ideal I of the polynomial ring is the set of polynomials that vanish to order n on the corresponding variety $V(I)$. Finding all these polynomials or even finding generators for the ideals $I^{(n)}$ is an extremely difficult task known geometrically as the Interpolation Problem.

Problem 3.2 (Higher Order Interpolation Problem)**.** *Given a finite set of points X in $\mathbb{P}^d$ and $n \in \mathbb{N}$ find the polynomials vanishing to order n on X.*

We will consider a less ambitious version of this problem.

Problem 3.3 (Least Degree Interpolation Problem)**.** *Given a finite set of points X in $\mathbb{P}^d$ and $n \in \mathbb{N}$ find the least degree of a non zero polynomial vanishing to order n on X.*

To talk about the least degree of a polynomial in a give ideal we use the following notation:

Notation 3.4. Let J be a graded ideal. The **initial degree** of J is

$$\alpha(J) = \min\{\deg(f) : 0 \neq f \in J\}.$$

Remark 3.5. A key property is that applying $\alpha()$ to a containment results in an opposite inequality. More precisely, if $I \subseteq J$ then $\alpha(I) \geq \alpha(J)$.

Exercise 8. Show that if $\{I^{\boxed{n}}\}_{n\in\mathbb{N}}$ is a **graded family of ideals** meaning that for all $a, b \in \mathbb{N}$

$$I^{\boxed{a}} I^{\boxed{b}} \subseteq I^{\boxed{a+b}}$$

then the sequence $\{\alpha_n := \alpha(I^{\boxed{n}})\}_{n\in\mathbb{N}}$ is subadditive meaning that for all $a, b \in \mathbb{N}$

$$\alpha_a + \alpha_b \geq \alpha_{a+b}.$$

Exercise 9. Show that if $\{\alpha_n\}_{n\geq 1}$ is a subadditive sequence then the limit below exists and is equal to the asserted infimum

$$\widehat{\alpha} := \lim_{n\to\infty} \frac{\alpha_n}{n} = \inf\left\{\frac{\alpha_n}{n} : n \geq 1\right\}.$$

In view of Exercise B.10 we define the following asymptotic invariant

Definition 3.6 (Waldschmidt [Wal77], Boci Harbourne[BH10a])**.** The **Waldschmidt constant** of a graded ideal I is the real number

$$\widehat{\alpha}(I) = \lim_{n\to\infty} \frac{\alpha(I^{(n)})}{n} = \inf\left\{\frac{\alpha(I^{(n)})}{n} : n \geq 1\right\}.$$

Definition 3.7. For a monomial ideal I, the **Newton polyhedron** of I, denoted $NP(I)$, is the convex hull of the exponent vectors for all the monomials in I

$$NP(I) = \operatorname{conv}\{(a_1, \ldots, a_d) \in \mathbb{N}^d \mid x_1^{a_1} \cdots x_d^{a_d} \in I\}.$$

Whenever we have a graded family of monomial ideals we can form an asymptotic Newton polyhedron that encodes the entire family. We give the definition for the particular case of the family of symbolic powers.

Definition 3.8. If I is a monomial ideal, the convex body

$$SP(I) = \bigcup_{n\geq 1} \frac{1}{n} NP\left(I^{(n)}\right).$$

is called the **symbolic polyhedron**.

Theorem 3.9 ([BCG+15])**.** *If I is a square-free monomial ideal with prime decomposition $I = P_1 \cap \cdots \cap P_r$, then the symbolic polyhedron of I is obtained as*

$$SP(I) = NP(P_1) \cap \cdots \cap NP(P_r)$$

and the Waldschmidt constant of I can be computed as

$$\widehat{\alpha}(I) = \min\{v_1 + \cdots + v_d : (v_1, \ldots, v_d) \text{ is a vertex of } SP(I)\}.$$

Example 3.10. The figure below shows a partial view of the facets of the Newton and symbolic polyhedra for the ideal $I = (xy, xz, yz)$ with prime decomposition $I = (x, y) \cap (x, z) \cap (y, z)$. The respective polyhedra are solid bodies located in the positive orthant and having the pictured facets as the outer boundary. In particular the vertices of $SP(I)$ are located at $(1, 1, 0), (1, 0, 1), (0, 1, 1), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Thus $\widehat{\alpha}(I) = \frac{3}{2}$.

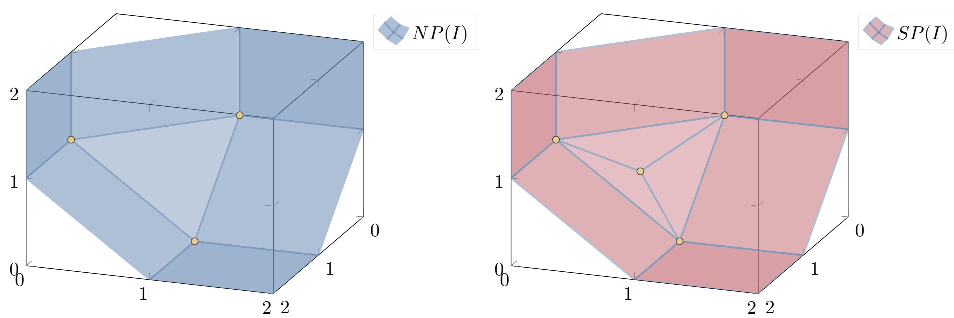

FIGURE 2. The Newton, symbolic and irreducible polyhedra of $I = (xy, xz, yz)$

In [Nag59] Nagata established the upper bound $\widehat{\alpha}(I(X)) \leq \sqrt{r}$ for any set X of $r \geq 9$ generic points in $\mathbb{P}^2$. Note that this upper bound also holds true for all sets of points (see, e.g, [Har77, Example 1.3.7]). Nagata also proposed, in different language, the following conjecture to the effect that very general sets of points attain the maximum value of the Waldschmidt constant permitted by this inequality.

Conjecture 3.11 (Nagata [Nag59])**.** *Any set X of $r \geq 10$ generic points in $\mathbb{P}^2$ over a field of characteristic zero satisfies $\alpha(I(X)^{(n)}) > n\sqrt{r}$ for all $n \in \mathbb{N}$. Equivalently, there is an equality*

$$\widehat{\alpha}(I(X)) = \sqrt{r}.$$

This statement holds true for r a perfect square, by Nagata's work in [Nag59], but it remains open for all other values of $r \geq 10$. We comment on the equivalence of the two claims in the above conjecture. For $\sqrt{r} \notin \mathbb{N}$ (the case that is still open), the conjectured inequality for initial degrees in Conjecture C.7 is equivalent to $\alpha(I(X)^{(n)}) \geq n\sqrt{r}$. Utilizing the known upper bound $\widehat{\alpha}(I_X) \leq \sqrt{r}$ and the description of the Waldschmidt constant as an infimum (see Definition 3.6), we see that the two statements in Conjecture C.7 are indeed equivalent.

Below we give further equivalent homological formulations of Nagata's conjecture. Intuitively, in homological terms this conjecture becomes the statement that the width of the Betti table of the symbolic powers of $I(X)$ grows sub-linearly.

Recall that the Castelnuovo-Mumford regularity of a graded module M is the maximum degree in which M has a nonzero Betti number $\operatorname{reg}(M) = \max\{j : \beta_{ij}(M) \neq 0\}$.

Definition 3.12. In the following we define the **asymptotic regularity** of a graded ideal I to be the following limit, when it exists:

$$\widehat{\operatorname{reg}}(I) = \lim_{n\to\infty} \frac{\operatorname{reg}(I^{(n)})}{n} = \inf\left\{\frac{\operatorname{reg}(I^{(n)})}{n} : n \geq 1\right\}.$$

See [Cut15] for an excellent survey regarding limits of this kind.

Conjecture 3.13 (Di Pasquale–Nguyễn–Seceleanu [DNS22])**.** *Any set X of $r \geq 10$ generic points in $\mathbb{P}^2$ over a field of characteristic zero satisfies*

$$\widehat{\alpha}(I(X)) = \widehat{\operatorname{reg}}(I(X)), \quad \textit{equivalently} \quad \lim_{n\to\infty} \frac{\operatorname{reg}(I(X)^{(n)}) - \alpha(I(X)^{(n)})}{n} = 0.$$

Iarrobino [Iar97] generalized Conjecture C.7 to projective spaces of arbitrary dimension.

Conjecture 3.14 (Iarrobino [Iar97])**.** *A set X of r generic points in the projective space $\mathbb{P}^N$ over a field of characteristic zero with $r \geq \max\{N+5, 2^N\}$ and $(r,N) \notin \{(7,2),(8,2),(9,3)\}$ satisfies $\alpha(I(X)^{(n)}) \geq n\sqrt[N]{r}$ for all $n \in \mathbb{N}$. Equivalently, apart from the given list of exceptions, there is an equality*

$$\widehat{\alpha}(I(X)) = \sqrt[N]{r}.$$

Conjecture 3.14 is known to hold only for the case $r = s^N$ by work of Evain [Eva05].

3.2. **Bounding the Waldschmidt Constant.** Beyond the setting of Conjecture C.7 it is an interesting problem to find bounds, if not exact values of the Waldschmidt constant of a graded ideal. Here we explore ways in which the Containment Problem 2.1 and its more general avatars with maximal ideal twist in Problem 3.16 can produce such bounds.

Before we do so we recall an important conjecture that arose in relation to Andrew Wiles's proof of Fermat's Lats Theorem.

Conjecture 3.15. *(Eisenbud–Mazur [EM97]). Let $(R,\mathfrak{m})$ be a regular local ring of characteristic zero. If $I \subseteq R$ is a radical ideal, then $I^{(2)} \subseteq \mathfrak{m}I$.*

Problem 3.16. *Fix a graded ideal I in a ring with graded maximal ideal $\mathfrak{m}$. For which triples $a,b,c \in \mathbb{N}$ does the containment*

$$I^{(a)} \subseteq \mathfrak{m}^c I^b$$

hold?

Lemma 3.17. *Let I be a graded ideal in a standard graded ring with graded maximal ideal $\mathfrak{m}$.*

(1) If I satisfies $I^{(sn)} \subseteq I^n$ for some $s \in \mathbb{N}$ and for all $n \gg 0$ then $\widehat{\alpha}(I) \geq \alpha(I)/s$.
(2) If I satisfies $I^{(sn)} \subseteq \mathfrak{m}^{(s-1)n} I^n$ for some $s \in \mathbb{N}$ and for all $n \gg 0$ then

$$\widehat{\alpha}(I) \geq (\alpha(I) + s - 1)/s. \tag{3.1}$$

(3) If I satisfies $I^{((s+m-1)n)} \subseteq \mathfrak{m}^{(s-1)n} \left(I^{(m)}\right)^n$ for some $s, m \in \mathbb{N}$ and for all $n \gg 0$ then

$$\widehat{\alpha}(I) \geq (\alpha(I^{(m)}) + s - 1)/(s + m - 1). \tag{3.2}$$

Proof. (1) Applying Remark 3.5 to $I^{(sn)} \subseteq I^n$ results in $\alpha(I^{(sn)}) \geq \alpha(I^n) = n\alpha(I)$. Dividing through by sn and taking limits yields the desired conclusion.

(2) Applying Remark 3.5 to $I^{(sn)} \subseteq \mathfrak{m}^{(s-1)n} I^n$ results in

$$\alpha(I^{(sn)}) \geq \alpha(\mathfrak{m}^{sn-n} I^n) = n\alpha(I) + sn - n.$$

Dividing through by sn yields

$$\frac{\alpha(I^{(sn)})}{sn} \geq \frac{n\alpha(I) + sn - n}{sn} = \frac{\alpha(I) + s - 1}{s} > \frac{\alpha(I)}{s}.$$

The desired conclusion is reached by passing to the limit.

The proof of (3) is similar. □

In particular, by Theorem 2.4, graded ideals in a d-dimensional polynomial ring satisfy (1) with $s = d - 1$. The corresponding bound on $\widehat{\alpha}$, although phrased in a different language, appears in work of Waldschmidt [Wal77] and Skoda [Sko77]. Improvements on this lower bound as given in (3.1) and (3.2) have been proposed by Chudnovsky [Chu81] and Demailly [Dem82], respectively, in relation to Problem 3.3. The validity of the bounds suggested by Chudnovsky and Demailly follows if one can establish the containments in Lemma 3.17. We make these containments precise below.

Question 3.18. Must a graded radical ideal I of big height h in a polynomial ring with maximal homogeneous ideal $\mathfrak{m}$ satisfy

$$I^{(hn)} \subseteq \mathfrak{m}^{(h-1)n} I^n \tag{3.3}$$

$$I^{((h+m-1)n)} \subseteq \mathfrak{m}^{(h-1)n} \left(I^{(m)}\right)^n \tag{3.4}$$

hold for all $m, n \geqslant 1$?

Remark 3.19. An affirmative answer to Question 3.18 implies Conjecture 2.19.

An affirmative answer to Question 3.18 (C.1) has been given for ideals defining general points in $\mathbb{P}^2$ in [HH13], for binomial numbers of generic sets of points in arbitrary projective spaces in [FMX18], for ideals defining generic sets of points in projective space $\mathbb{P}^d$ of cardinality at least 2^d in [DTG17], and for ideals defining sufficiently many general points in projective space and $n \gg 0$ in [BGHN22a]. The inequality in Lemma 3.17 (3) is proven for general points in $\mathbb{P}^2$ by Esnault and Viehweg [EV83] and for generic sets of projective points of sufficiently large cardinality in arbitrary projective spaces by work of Malara, Szemberg and Szpond [MSS18], extended by Chang and Jow [CJ20]. For sufficiently large general sets of points in arbitrary projective spaces the same inequality follows from work of Bisui, Grifo, Hà and Nguyễn in [BGHN22b], where an affirmative answer to Question 3.18 (C.2) is also provided in the same context for infinitely many values of n, although not for all n or even $n \gg 0$.

Outside the context of radical ideals, Question 3.18 can sometimes be answered in the negative.

Example 3.20 (Hoefel)**.** Let $I = (xy^2, yz^2, zx^2, xyz) = (x^2, y) \cap (y^2, z) \cap (z^2, x) \subset K[x, y, z]$, an ideal of height $h = 2$. When $n = 2$, Question 3.18 (1) suggests that $I^{(3)} \subseteq \mathfrak{m}I^2$. The monomial $x^2y^2z^2$ is in $I^{(3)} = (x^2, y) \cap (y^2, z) \cap (z^2, x)$. Although $x^2y^2z^2 \in I^2$, we do not have $x^2y^2z^2 \in \mathfrak{m}I^2$. Thus $I^{(3)} \not\subseteq \mathfrak{m}I^2$.

For square-free monomial ideals, Question 3.18 has an affirmative answer by [CEHH17]. The most general statement regarding containments for such ideals is:

Theorem 3.21 (Cooper Embree Ha Hoefel [CEHH17])**.** *Suppose I is a square-free monomial ideal of big heigh h. Then for all positive integers m, n and r we have the containment*

$$I^{((h+m-1)n-h+r)} \subseteq \mathfrak{m}^{(h-1)(n-1)+r-1}\left(I^{(m)}\right)^n.$$

3.3. **Resurgence.** The various invariants defined below under the name of resurgence were introduced to study the *containment problem* which asks for pairs of natural numbers d, n for which $I^{(d)} \subseteq I^n$.

Definition 3.22 (Boci Harbourne [BH10b])**.** The **resurgence** of an ideal I is the quantity

$$\rho(I) = \sup\left\{\frac{d}{n} : I^{(d)} \not\subseteq I^n\right\}.$$

and the **asymptotic resurgence** is

$$\widehat{\rho}(I) = \sup\left\{\frac{d}{n} : I^{(dt)} \not\subseteq I^{nt} \text{ for } t \gg 0\right\}.$$

By definition we have $\widehat{\rho}(I) \leq \rho(I)$. Exercise B.8 gives that if h is the big height of I then $\rho(I) < h$, a property that is called having **expected resurgence**, implies the stable Harbourne Conjecture 2.19. In fact even the seemingly weaker inequality $\widehat{\rho}(I) < h$ implies the stable Harbourne Conjecture 2.19 [HKZ22]. Ideals with respected resurgence include:

- square-free monomial ideals [DD21],
- ideals defining general points in $\mathbb{P}^d$ [BGHN22b],
- $P \subseteq K[x, y, z]$ a defining ideal for a semigroup ring $K[t^a, t^b, t^c]$ [FI22],
- ideals I in a local or graded ring with (grade) maximal ideal $\mathfrak{m}$ such that $I^{(n)} = I^n : \mathfrak{m}$ for all $n \in \mathbb{N}$ which satisfy $I^{(hn-h+1)} \subseteq \mathfrak{m}I^n$ for some fixed $n \geq 1$ [GHM20],
- in positive characteristic, ideals I such that R/I is Gorenstein [GHM23].

Useful bounds on resurgence were given by Boci–Harbourne and Guardo–Harbourne–Van Tuyl. In many situations , for example when $\dim(R/I) = 1$ and $\alpha(I) = \operatorname{reg}(I)$ these bounds allow to compute the resurgence of I without explicitly answering the Containment Problem 2.1.

Theorem 3.23 (Boci–Harbourne [BH10a], Guardo–Harbourne–Van Tuyl [GHVT13])**.**

(1) If I is a graded ideal then of big height h then

$$\frac{\alpha(I)}{\widehat{\alpha}(I)} \leq \widehat{\rho}(I) \leq \rho(I) \leq h.$$

(2) If I is the ideal such that $I^{(n)} = I^n : \mathfrak{m}^\infty$ for all $n \geq 1$ then

$$\widehat{\rho}(I) \leq \frac{\omega(I)}{\widehat{\alpha}(I)} \leq \frac{\operatorname{reg}(I)}{\widehat{\alpha}(I)},$$

where $\omega(I)$ denotes the least degree of a minimal generator of I.

4. A Zoo of Examples and Counterexamples

Even if we start with a prime ideal, computing the symbolic powers is difficult.

Example 4.1. Fix a field k, and let $R = k[x, y, z]$. Consider the ideal P given by

$$P = \Big(\underbrace{x^3 - yz}_{f}, \underbrace{y^2 - xz}_{g}, \underbrace{z^2 - x^2y}_{h}\Big).$$

There is a ring isomomorphism

$$\frac{k[x,y,z]}{P} \xrightarrow{\ \pi\ } k[t^3, t^4, t^5] .$$

$$(x, y, z) \longmapsto (t^3, t^4, t^5)$$

Since $k[t^3, t^4, t^5] \subseteq k[t]$ is a domain, we conclude that P is a prime ideal. In fact, P is a homogeneous ideal with the grading $\deg(x) = 3, \deg(y) = 4, \deg(z) = 5$ we considered above: our generators f, g, and h are now homogeneous, with $\deg(f) = 9$, $\deg(g) = 8$, and $\deg(h) = 10$. We claim that $P^{(2)} \neq P^2$.

Consider the homogeneous element $f^2 - gh \in (x)$, which has degree 18, and let q be such that $f^2 - gh = qx$. Since $x \notin P$ and $xq = f^2 - gh \in P^2$, we conclude that $q \in P^{(2)}$. However, since $\deg(x) = 3$ and $\deg(f^2) = 18$, q must be a homogeneous element of degree 15, but the smallest degree of any element in P^2 is $2 \times 8 = 16$, so $q \notin P^2$.

Example 4.2. Consider a 3×3 matrix of variables,

$$X = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix}.$$

Given a field k, we write $R = k[X]$ for the polynomial ring $R = k[x_1, \dots, x_9]$, and $I_t(X)$ for the ideal in R generated by the t-minors of X. Let $I = I_2(X)$, which is a homogeneous ideal generated in degree 2. It is a nontrivial fact (which we won't prove) that this ideal is in fact prime. We claim $I^{(2)} \neq I^2$, and once more we will use a degree argument. To see this, consider the element $\det(X)$, which we can write for example using cofactor expansion on the first row:

$$\det(X) = x_1 \begin{vmatrix} x_5 & x_6 \\ x_8 & x_9 \end{vmatrix} - x_2 \begin{vmatrix} x_4 & x_6 \\ x_7 & x_9 \end{vmatrix} + x_3 \begin{vmatrix} x_4 & x_5 \\ x_7 & x_8 \end{vmatrix}.$$

This is clearly an element in I, since we wrote it as a linear combination of elements in I. On the other hand, this is *not* an element in I^2, since it has degree 3 and $\alpha(I^2) = 4$. On the other hand, with a few careful computations one can see that

$$x_1 \det(X) = \begin{vmatrix} x_1 & x_3 \\ x_7 & x_9 \end{vmatrix} \begin{vmatrix} x_1 & x_2 \\ x_4 & x_5 \end{vmatrix} - \begin{vmatrix} x_1 & x_2 \\ x_7 & x_8 \end{vmatrix} \begin{vmatrix} x_1 & x_3 \\ x_4 & x_6 \end{vmatrix} \in I^2,$$

and since $x_1 \notin I$, we conclude that $\det(X) \in I^{(2)}$. So we have shown that $I^2 \neq I^{(2)}$.

The assumption that R is regular is necessary in Lemma 1.17; we cannot extend this result to any noetherian local ring.

Example 4.3. When $R = k[\![x, y, z]\!]/(xy - z^c)$ and $c \geq 2$, the prime $P = (x, z)$ satisfies $x \in P^{(c)}$, so in particular $P^{(c)} \nsubseteq \mathfrak{m}^c$. One can show that $P^{(cn)} = (x^n)$, so in fact $P^{(cn)} \subseteq \mathfrak{m}^n$ for all $n \geqslant 1$.

We also cannot weaken the perfect hypothesis in Theorem 1.22.

Example 4.4. Let $K = \mathbb{F}_p(t)$, with p prime, and consider the ring $R = K[x]$ and the prime ideal $P = (x^p - t)$. As we described in Example 1.19, $D^1_{R|K} = R \oplus R\frac{\partial}{\partial x}$, and $\frac{\partial}{\partial x}(x^p - t) = 0 \in P$. Therefore, $P^{\langle 2 \rangle} = P$. On the other hand, P is a principal ideal in a domain, so its symbolic powers are the powers; in particular, $P^{(2)} = P^2 \neq P^{\langle 2 \rangle}$.

Appendix A. Scripts for Macaulay2

Macaulay2 [GS] s a software system for commutative algebra and algebraic geometry.

Running macaulay 2 online: if you do not have Macaulay2 installed, you can run it in your browser at https://www.unimelb-macaulay2.cloud.edu.au/#home.

Entering ideals and methods dealing with primes:

- The method `minimalPrimes` receives an ideal and returns a list of its minimal primes.

```
i1 : R = QQ[x,y];

i2 : I = ideal {x^2,x*y};

i3 : minimalPrimes I

o3 = {ideal x}

o3 : List
```

- Given an R-module M, `associatedPrimes M` or `ass` will return a list, the list of all the primes in $\mathrm{Ass}_R(M)$. If I is an ideal in R, Macaulay2 follows the same convention we do: `associatedPrimes I` will return $\mathrm{Ass}(I)$, the associated primes of the module R/I.

```
i1 : R = QQ[x,y];

i2 : I = ideal {x^2,x*y};

i3 : associatedPrimes I

o3 = {ideal x, ideal (y, x)}

o3 : List
```

Entering semigroup rings

- Here is a routine that creates a prime ideal P so that for some given integers a, b, c, $k[t^a, t^b, t^c] \cong k[x, y, z]/P$. In the line `i8` we ask for the degrees of the minimal generators of P.

```
i1 : k = QQ;

i2 : a = 9; b = 11; c = 14;

i6 : R = k[x,y,z, Degrees => {a,b,c}];
```

```
i7 : P = ker map(k[t],R,{t^a,t^b,t^c})
               4    2      3    3   5    3 2
o7 = ideal (x  - y z, x*y  - z , y  - x z )

o7 : Ideal of R

i8: degrees P

o8: {{36}, {42}, {55}}
```

The Symbolic Powers Package written by Eloísa Grifo is a great tool for working with these ideals.

- The main method in the `SymbolicPowers` package is `symbolicPower`, which takes as inputs an ideal I and an integer n and returns $I^{(n)}$.

```
i1 : loadPackage "SymbolicPowers";

i2 : R=QQ[x,y,z];

i3 : I=ideal(x*(y^3-z^3),y*(z^3-x^3),z*(x^3-y^3));
o3 : ideal of R

i4 : transpose mingens symbolicPower(I,2)
o4: {-8} | x3y3z2-y6z2-x3z5+y3z5 |
    {-8} | x6z2-y6z2-2x3z5+2y3z5 |
    {-8} | x4y3z-xy6z-x4z4+xy3z4 |
    {-8} | x6yz-x3y4z-x3yz4+y4z4 |
    {-8} | x2y6-2x2y3z3+x2z6     |
    {-8} | x3y5-x3y2z3-y5z3+y2z6 |
    {-8} | x4y4-x4yz3-xy4z3+xyz6 |
    {-8} | x5y3-x5z3-x2y3z3+x2z6 |
    {-8} | x6y2-2x3y2z3+y2z6     |
```

- Using `containmentProblem`, the user can determine the smallest value of a, given b, for which $I^{(a)} \subseteq I^b$. We can ask the same question backwards: given a, what is the largest power b such that $I^{(a)} \subseteq I^b$? To ask the latter question one makes the optional parameter `InSymbolic` true as illustrated on input line 6 below.

```
i1 : loadPackage "SymbolicPowers";

i2 : R=QQ[x,y,z];

i3 : I=ideal(x*(y^3-z^3),y*(z^3-x^3),z*(x^3-y^3));
o3 : ideal of R

i4 : containmentProblem(I,2)
o4 : 4
```

```
i6 : containmentProblem(I,5, InSymbolic=>true)
o6 : 3
```

- Our package computes Waldschmidt constants of monomial ideals by finding their `symbolicPolyhedron`.

```
i1 : loadPackage "SymbolicPowers";

i2 : R=QQ[x,y,z];

i3 : I=ideal(x*y,x*z,y*z);

i4 : symbolicPolyhedron(I)
o4 = {ambient dimension => 3           }
      dimension of lineality space => 0
      dimension of polyhedron => 3
      number of facets => 6
      number of rays => 3
      number of vertices => 4
o4 : Polyhedron

i5 : waldschmidt I
Ideal is monomial, the Waldschmidt constant is computed exactly

      3
o5 =  -
      2
o5 : QQ
```

Appendix B. List of Exercises

Exercise B.1. Using Macaulay2 or another computer algebra system find the prime ideal P so that $k[x,y,z]/P \cong k[t^3,t^4,t^5]$. Is $P^{(2)} = P^2$?

Exercise B.2. In this exercise we consider the question: if $I^{(n)} = I^n$, must $I^{(n+1)} = I^{n+1}$?

Here is an example due to Susan Morey. Consider a 3×3 matrix of variables,

$$X = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix}$$

and let I be the ideal generated by the 2×2 minors of X. Is $I^2 = I^{(2)}$? Is $I^3 = I^{(3)}$?

You may use Macaulay2 or another computer algebra system to answer the questions. In Macaulay 2, I can be entered as follows:

```
i1 : R=QQ[x_1..x_9];

i2 : X=genericMatrix(R,3,3);

i3: I=minors(2,X)
```

Equality of I and J is checked in Macaulay2 using `I==J`.

Exercise B.3. We have seen in the proof of the Zariski–Nagata theorem that in a regular ring R if P is a prime ideal and $\mathfrak{m}$ is a maximal ideal such that $P \subseteq \mathfrak{m}$ then $P^{(n)} \subseteq \mathfrak{m}^n$ for all $n \geq 1$. The following shows that the assumption R regular is needed. The last question shows that the assumption R regular is also needed in the Eisenbud-Mazur conjecture which we will discuss later.

Let $R = k[\![x, y, z]\!]/(xy - z^c)$ and $c \geq 2$. For the prime ideal $P = (x, z)$ verify that $x \in P^{(c)}$, so in particular $P^{(c)} \nsubseteq \mathfrak{m}^c$. For $c = 2$, is $P^{(2)} \subseteq \mathfrak{m}P$?

Exercise B.4. Let $R = k[x_1, \dots, x_d]$, I be an ideal of R, and n be a positive integer. Prove that for each integer $n \geq 1$ the following set (the n-th differential power of I) is an ideal

$$I^{\langle n \rangle} := \left\{ f \in R \mid \frac{\partial^{a_1 + \cdots + a_d} f}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}} \in I \text{ for all } a_i \geq 0, a_1 + \cdots + a_d \leq n \right\}.$$

Exercise B.5. In the proof of the Zariski-Nagata theorem we stated a Lemma that says that if K is perfect and I is an ideal of $K[x_1, \dots, x_d]$ then $I^{(n)} = I^{\langle n \rangle}$. We show that K perfect is needed.

Let $K = \mathbb{F}_p(t)$, with p prime and $\mathbb{F}_p$ the field with p elements. Consider the ring $R = K[x]$ and the prime ideal $P = (x^p - t)$. Find the second differential power $P^{\langle 2 \rangle}$ and the second symbolic power $P^{(2)}$. Are these ideals equal: $P^{\langle 2 \rangle} = P^{(2)}$?

Exercise B.6. Use the pigeonhole principle to show that if I is an ideal generated by monomials in a polynomial ring then Harbourne's Conjecture holds.

Exercise B.7. Let $R = k[x_1, \dots, x_d]$ and consider the squarefree monomial ideal

$$I = \bigcap_{i<j} (x_i, x_j).$$

Show that while $I^{(2n-1)} \subseteq I^n$ holds for all $n \geqslant 1$, $I^{(2n-2)} \nsubseteq I^n$ for $n < d$, so we cannot do better than Harbourne's Conjecture in this case. What happens when $n = d$? How does this example generalize to higher height?

Exercise B.8. Prove that the stable Harbourne holds

(1) if $I^{(hm-h)} \subseteq I^m$ holds for some value m
(2) if $I^{(hm-h+1)} \subseteq I^m$ holds for some m and $I^{(n+h)} \subseteq II^{(n)}$ holds for all $n > m$
(3) if $\sup\{\frac{a}{b} : I^{(a)} \nsubseteq I^b\} < h$.

Exercise B.9. For a homogeneous ideal J of $R = k[x_1, \dots, x_d]$ denote

$$\alpha(J) = \min\{\deg(f) : 0 \neq f \in J\}.$$

Show that if $\{I^{\boxed{n}}\}_{n \in \mathbb{N}}$ is a graded family of homogeneous ideals meaning that for all $a, b \in \mathbb{N}$

$$I^{\boxed{a}} \cdot I^{\boxed{b}} \subseteq I^{\boxed{a+b}}$$

then the sequence $\{\alpha_n := \alpha(I^{\boxed{n}})\}_{n \in \mathbb{N}}$ is subadditive meaning that for all $a, b \in \mathbb{N}$

$$\alpha_a + \alpha_b \geq \alpha_{a+b}.$$

Exercise B.10. Show that if $\{\alpha_n\}_{n\geq 1}$ is a subadditive sequence then the limit below exists and is equal to the asserted infimum

$$\widehat{\alpha} := \lim_{n\to\infty} \frac{\alpha_n}{n} = \inf\left\{\frac{\alpha_n}{n} : n \geq 1\right\}.$$

Appendix C. List of Open Problems

Problem C.1. *(Containment Problem). Let I be an ideal in a noetherian ring R. For what values of a and b is $I^{(a)} \subseteq I^b$?*

Question C.2 (Huneke, 2000)**.** Let P be a prime ideal of height 2 in a regular local ring R. Is $P^{(3)} \subseteq P^2$?

Conjecture C.3 (Stable Harbourne conjecture by Grifo [Gri20])**.** *Let I be a radical ideal of big height h in a regular ring R. Then $I^{(hn-h+1)} \subseteq I^n$ for all n sufficiently large.*

Conjecture C.4. *(Eisenbud–Mazur [EM97]). Let $(R, \mathfrak{m})$ be a regular local ring of characteristic zero. If $I \subseteq R$ is a radical ideal, then $I^{(2)} \subseteq \mathfrak{m}I$.*

Problem C.5. *Fix a graded ideal I in a ring with graded maximal ideal $\mathfrak{m}$. For which triples $a, b, c \in \mathbb{N}$ does the following containment hold:*

$$I^{(a)} \subseteq \mathfrak{m}^c I^b \ ?$$

Question C.6. Must a graded radical ideal I of big height h in a polynomial ring with maximal homogeneous ideal $\mathfrak{m}$ satisfy

$$I^{(hn)} \subseteq \mathfrak{m}^{(h-1)n} I^n \tag{C.1}$$

$$I^{((h+m-1)n)} \subseteq \mathfrak{m}^{(h-1)n} \left(I^{(m)}\right)^n \tag{C.2}$$

hold for all $m, n \geqslant 1$?

Conjecture C.7 (Nagata [Nag59])**.** *Any set X of $r \geq 10$ generic points in $\mathbb{P}^2$ over a field of characteristic zero satisfies $\alpha(I(X)^{(n)}) > n\sqrt{r}$ for all $n \in \mathbb{N}$. Equivalently, there is an equality*

$$\widehat{\alpha}(I(X)) = \sqrt{r}.$$

Conjecture C.8 (DiPasquale–Nguyễn–Seceleanu [DNS22])**.** *Any set X of $r \geq 10$ generic points in $\mathbb{P}^2$ over a field of characteristic zero satisfies*

$$\widehat{\alpha}(I(X)) = \widehat{\operatorname{reg}}(I(X)), \qquad \textit{equivalently} \qquad \lim_{n\to\infty} \frac{\operatorname{reg}(I(X)^{(n)}) - \alpha(I(X)^{(n)})}{n} = 0.$$

Conjecture C.9 (Iarrobino [Iar97])**.** *A set X of r generic points in the projective space $\mathbb{P}^N$ over a field of characteristic zero with $r \geq \max\{N+5, 2^N\}$ and $(r, N) \notin \{(7,2), (8,2), (9,3)\}$ satisfies $\alpha(I(X)^{(n)}) \geq n\sqrt[N]{r}$ for all $n \in \mathbb{N}$. Equivalently, apart from the given list of exceptions, there is an equality*

$$\widehat{\alpha}(I(X)) = \sqrt[N]{r}.$$

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