Math 953 Notes

Spring 2025

Contents

Conventions

I am assuming knowledge of commutative rings at the level of an introductory graduate course in *Commutative Algebra*, e.g. UNL's Math 905.

We stipulate that throughout the course

- all rings are commutative, unital, and nontrivial, meaning that $0 \neq 1$.
- all ring homomorphisms $R \to S$ are assumed to map $1_R \mapsto 1_S$
- we will work over an algebraically closed field k, unless specified otherwise
- $k[x_1, \ldots, x_n]$ denotes a polynomial ring in n variables $x_1, \ldots, x_n$ with coefficients in k
- given $f \in k[x_1, \ldots, x_n]$ and $\mathbf{a} = (a_1, \ldots, a_n) \in k^n$, $f(\mathbf{a}) = f(a_1, \ldots, a_n)$
- given $f \in k[x_1, \ldots, x_n]$ written as a finite sum $f = \sum \alpha_{\mathbf{m}} x_1^{m_1} \cdots x_n^{m_n}$ with $\alpha_{\mathbf{m}} \in k$, we denote
$$\deg(f) = \max\{m_1 + \cdots + m_n \mid \alpha_{\mathbf{m}} \neq 0\}$$
- given $\mathbf{a} = (a_1, \ldots, a_n) \in k^n$, $[\mathbf{a}] \in \mathbb{P}_k^n$ denotes the equivalence class of $\mathbf{a}$, that is, the set $[\mathbf{a}] = \{\lambda \mathbf{a} : 0 \neq \lambda \in k\}$

Chapter 1

Affine varieties

Wednesday, January 22

What is this course about?

In this course we will study systems of polynomial equations in several variables.

Many sets have a natural description as solutions of such polynomial systems.

Example 1.1. Consider the set X of 3×4 matrices with entries in $\mathbb{C}$ of rank at most two. We can think of each such matrix as a point in $\mathbb{C}^{12}$ via the bijective correspondence

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} \leftrightarrow \mathbf{a} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{21} & a_{22} & \cdots & a_{34}. \end{pmatrix}$$

Observe that

$$X = \{A \in \mathrm{M}_{3\times 4}(\mathbb{C}) \mid \Delta_1(A) = 0, \Delta_2(A) = 0, \Delta_3(A) = 0, \Delta_4(A) = 0\},$$

where $\Delta_1, \ldots \Delta_4$ are the 3×3 minors of A which are *polynomials* in the a_{ij}. Thus X is the set of solutions to a system of 4 polynomial equations of degree 3 in 12 variables. As we will soon see, this makes X a *variety*.

We can ask:

- *What is the dimension of X?* At this point use your intuition from linear algebra for what dimension may mean, stipulating that the dimension of the solution space of a system of equations is the number of *free variables* we need to write down the general solution.

  I claim that the dimension of X is 10. Why? Say $A \in X$. Because the rank of A is at most 2, we can find $\alpha, \beta \in \mathbb{C}$ so that

  $$\begin{pmatrix} a_{31} & a_{32} \end{pmatrix} = \alpha \begin{pmatrix} a_{11} & a_{12} \end{pmatrix} + \beta \begin{pmatrix} a_{21} & a_{22} \end{pmatrix}$$

and moreover, the values of α, β are determined by the six a_{ij} appearing in the equation above. We also have

$$\begin{pmatrix} a_{33} & a_{34} \end{pmatrix} = \alpha \begin{pmatrix} a_{13} & a_{14} \end{pmatrix} + \beta \begin{pmatrix} a_{23} & a_{24} \end{pmatrix},$$

so $a_{3,3}$ and $a_{3,4}$ are determined by the other 10 entries of A, which are free variables.

- *What does X look like? For example, how many components does it have?*
- *Is X smooth or singular?*
- *Does X have a simpler embedding somewhere other than $\mathbb{C}^{12}$?*
- *How would X intersect with another set, say a 2-dimensional plane in $\mathbb{C}^{12}$?*

1.1 Affine varieties

In this subsection, k is an arbitrary field.

Definition 1.2. For an arbitrary field k, *affine n-space* over k, written $\mathbb{A}^n_k$, is the set of n-tuples of elements of k:

$$\mathbb{A}^n_k = \{(a_1, \dots, a_n) \mid a_i \in k\}.$$

So, $\mathbb{A}^n_k$ is just k^n, but we avoid using the latter since k^n is the usual notation of the standard n-dimensional k-vector space. Affine space is not thought of as a vector space typically.

Recall $k[x_1, \dots, x_n]$ is the ring of all polynomials in the n variables $x_1, \dots, x_n$ having k coefficients. For any subset S of $k[x_1, \dots, x_n]$, the *zero locus* of S, written $V(S)$, is the subset of $\mathbb{A}^n_k$ giving the common zeroes of all the members of S; that is,

$$V(S) := \{(a_1, \dots, a_n) \in \mathbb{A}^n_k \mid f(a_1, \dots, a_n) = 0, \text{ for all } f \in S\} \subseteq \mathbb{A}^n_k.$$ [1]

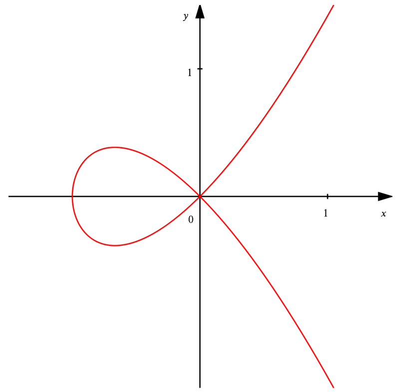
When we need to indicate the ground field, we write $V_k(S)$ for $V(S)$. Also, given a finite list $f_1, \dots, f_m \in k[x_1, \dots, x_n]$, by $V(f_1, \dots, f_m)$ we mean $V(\{f_1, \dots, f_m\})$.

Definition 1.3. An *affine variety* is a subset of $\mathbb{A}^n_k$, for some n, that is equal to $V(S)$ for some subset S of $k[x_1, \dots, x_n]$. Many people call these sets *algebraic sets* and reserve the terminology affine varieties for the irreducible algebraic sets (to be defined).

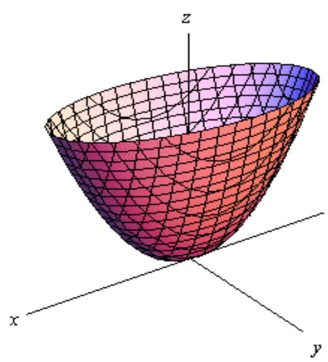
Here are some pictures of affine varieties.

- For $k = \mathbb{R}$ and $n = 2$, $V(y^2 - x^2(x+1))$ is a *nodal curve* in $\mathbb{A}^2_{\mathbb{R}}$, the real plane. Note that we've written x for x_1 and y for x_2 here.

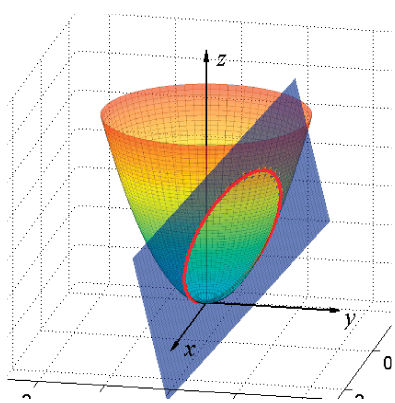
[1] There is a good pedagogical reason why this set was denoted $Z(S)$ in Math 905. However, the notation here is more standard and the danger of potential confusion does not exist in this course.



- For $k = \mathbb{R}$ and $n = 3$, $V(z - x^2 - y^2)$ is a paraboloid in $\mathbb{A}^3_{\mathbb{R}}$, real three space. Note that $x = x_1$, $y = x_2$ and $z = x_3$.



- For $k = \mathbb{R}$ and $n = 3$, $V(z - x^2 - y^2, 3x - 2y + 7z - 7)$ is an elipse in $\mathbb{A}^3_{\mathbb{R}}$.



- **The field matters:** $V_{\mathbb{R}}(x^2 + y^2 + 1) = \emptyset$, while $V_{\mathbb{C}}(x^2 + y^2 + 1) \neq \emptyset$.

Here are some important classes of affine varieties.

Example 1.4.

- $\mathbb{A}^n_k = V((0)$ is an affine variety;
- $\emptyset = V((1))$ is an affine variety;

- Any singleton $\{(a_1, \ldots, a_n)\}$ is an affine variety, since it is the zero locus of the set $\{x_1 - a_1, \ldots, x_n - a_n\}$ and so is any finite set of points in $\mathbb{A}^n$;
- Linear subspaces of k^n are affine varieties;
- Given $0 \neq f \in k[x_1, \ldots, x_n]$, the affine variety $V(f)$ is called a *hypersurface*;
- Given a linear polynomial $\ell = c_1x_1 + \cdots c_nx_n + c_0$ with $c_i \in k$, the affine variety $V(\ell)$ is called a *hyperplane*;
- If $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ are affine varieties, so is their Cartesian product $X \times Y = \{(x, y) \mid x \in X, y \in Y\} \subseteq \mathbb{A}^m \times \mathbb{A}^n = \mathbb{A}^{m+n}$.

Exercise 1.5. Let $V = \{(t^2, t^3) \mid t \in k\} \subseteq \mathbb{A}^2_k$. Prove V is an affine variety.

Exercise 1.6. Let $V = \{(t, t^2, t^3) \mid t \in k\} \subseteq \mathbb{A}^3_k$. Prove V is an affine variety. This is called the (affine) *twisted cubic* curve.

Exercise 1.7. Identifying $n \times n$ matrices with points in $\mathbb{A}^{n^2}_k$, prove that the set $SL_n(k)$ is an affine variety and that $GL_n(k)$ is an affine variety if and only if k is finite.

Exercise 1.8. Let $V = \mathbb{Z}^n \subseteq \mathbb{A}^n_k$, where k is a field of characteristic zero and thus $\mathbb{Z} \subseteq k$. Prove V is not an affine variety.

Exercise 1.9. Let $V = \{(t, \sin(t)) \mid t \in \mathbb{R}\} \subseteq \mathbb{A}^2_{\mathbb{R}}$. Prove V is not an affine variety.

Exercise 1.10. Prove that any finite set of points of $\mathbb{A}^n_k$ for any $n \geq 1$ is an affine variety.

Exercise 1.11. Is every countably infinite set of points in $\mathbb{A}^n_{\mathbb{C}}$ an affine variety?

Exercise 1.12. Show $S = \mathbb{A}^2_k \setminus \{(0,0)\} = \{(a, b) \in \mathbb{A}^2_k \mid (a, b) \neq (0, 0)\}$ is not an affine variety whenever k is an infinite field. Is this true when k is a finite field?

Friday, January 24

Here are some easy-to-prove properties of affine varieties.

Proposition 1.13. *For any fixed n:*

1. *If $S_1 \subseteq S_2 \subseteq k[x_1, \ldots, x_n]$ then $V(S_2) \subseteq V(S_1)$.*
2. *If $S_1, S_2 \subseteq k[x_1, \ldots, x_n]$ then $V(S_1) \cup V(S_2) = V(S_1 \cdot S_2)$ where $S_1 \cdot S_2$ is the set of products of elements of the form ab with $a \in S_1$ and $b \in S_2$. In particular, a finite union affine varieties in $\mathbb{A}^n_k$ is again an affine variety and by induction the union of a finite number of affine varieties in $\mathbb{A}^n_k$ is again an affine variety.*

3. *If* $\{S_\alpha\}_{\alpha\in A}$ *is any collection of subsets of* $k[x_1,\dots,x_n]$*, then*

$$\bigcap_{\alpha\in A} V(S_\alpha) = V(\bigcup_{\alpha\in A} S_\alpha).$$

In particular, an arbitrary intersection of affine varieties in $\mathbb{A}^n$ *is again an affine variety.*

Proof. I'll just prove the second one: If $P \in V(S_1)\cup V(S_2)$ then for all $f \in S_1$, $g \in S_2$ we have $(fg)(P) = f(P)g(P) = 0$ since either $f(P) = 0$ or $g(P) = 0$. Thus $P \in V(\{fg\} \mid f \in S_1, g \in S_2\}) = V(S_1S_2)$.

Say $P \notin V(S_1)\cup V(S_2)$. Then for any $f \in S_1, g \in S_2$, we have $f(P) \neq 0$ and $g(P) \neq 0$ and thus $(fg)(P) = f(P)g(P) \neq 0$. This proves $P \notin V(\{fg\} \mid f \in S_1, g \in S_2\}) = V(S_1S_2)$.

We conclude that $V(S_1)\cup V(S_2) = V(S_1\cdot S_2)$. □

For a subset S of $k[x_1,\dots,x_n]$, let (S) denote the *ideal* of this ring *generated by* S; that is,

$$(S) = \{f_1g_1 + \cdots + f_mg_m \mid m \geq 0, f_i \in k[x_1,\dots,x_n], g_i \in S, \text{ for all } i\}.$$

For any ideal I, we let $\sqrt{I} = \{f \in k[x_1,\dots,x_n] \mid f^m \in I, \text{ for some } m \geq 1\}$, the *radical of* I. The radical of an ideal is again an ideal.

Proposition 1.14. *For any subset* S *of* $k[x_1,\dots,x_n]$*, if we set* $J = (S)$*, we have*

$$V(S) = V(J) = V(\sqrt{J}).$$

In particular, $V(J) = V(\sqrt{J})$ *for any ideal* J*.*

Proof. The containments $\supseteq$ hold since $S \subseteq (S) \subseteq \sqrt{(S)}$. Say $\mathbf{a} \in V(S)$. An arbitrary element of (S) has the form $f = f_1g_1 + \cdots + f_mg_m$ with $f_i \in k[x_1,\dots,x_n]$ and $g_i \in S$ for all i. We have $f(\mathbf{a}) = \sum_i f_i(\mathbf{a})g_i(\mathbf{a}) = \sum_i f_i(\mathbf{a})\cdot 0 = 0$. This proves $V(S) = V((S))$. Given any ideal I, if $\mathbf{a} \in V(J)$ and f is such that $f^n \in J$, then $f(\mathbf{a})^n = (f^n)(\mathbf{a}) = 0$ and hence $f(\mathbf{a}) = 0$. This proves $V(J) = V(\sqrt{J})$. □

Thanks to this proposition, we can redefine an affine variety to be a subset of the form $V(J)$ where J is an ideal (or even a radical ideal).

Exercise 1.15. Prove that if I, J are ideals of $k[x_1,\dots,x_n]$ then

1. $V(I + J) = V(I)\cap V(J)$;
2. $V(IJ) = V(I)\cup V(J)$.

Recall from Math 905:

Theorem 1.16 (Hilbert Basis Theorem)**.** *The ring* $k[x_1,\dots,x_n]$ *is noetherian. That is, every ideal of it is finitely generated or, equivalently, the collection of all ideals of this ring has the ascending chain condition.*

We won't prove this Theorem in this class; I assume you've seen it before.

Corollary 1.17. *For any field k, every affine variety V in $\mathbb{A}^n_k$ is the zero locus of a* finite *list of polynomials.*

Proof. By Proposition 1.14 we may assume $V = V(I)$ for a (radical) ideal I. By Theorem 1.16 we have $I = (g_1, \ldots, g_m)$ for some g_i's and hence (using Proposition 1.14 again) $V = V(g_1, \ldots, g_m)$. □

1.2 Regular maps

Having introduced affine varieties, we are now concerned with what are the right kind of morphisms between them.

1.2.1 Coordinate rings

The first case to think about are the scalar-valued maps, that is, functions from an affine variety $X \subseteq \mathbb{A}^n_k$ to $\mathbb{A}^1_k = k$. The natural algebraic maps from $\mathbb{A}^n_k$ to $\mathbb{A}^1_k = k$ are the polynomials $f(x_1, \ldots, x_n)$, that is, the elements of the ring $k[x_1, \ldots, x_n]$. We would like the inclusion $\iota : X \hookrightarrow \mathbb{A}^n_k$ to be a morphism and we would also like that the composition of two morphisms be a morphism. From all this it follows that the restriction of a polynomial function on $\mathbb{A}^n_k$ to $X \subseteq \mathbb{A}^n_k$ should be a morphism $f : X \to \mathbb{A}^1_k$. We declare that these are all the scalar-valued morphisms.

Definition 1.18. Let $X \subseteq \mathbb{A}^n_k$ be an affine variety. A function $F : X \to \mathbb{A}^1_k$ is *regular* if there exists $f \in k[x_1, \ldots, x_n]$ such that $F = f|_X$, that is, $f(\mathbf{a}) = F(\mathbf{a})$ for all $\mathbf{a} \in X$.

The *ring of regular functions* (or the *coordinate ring*) of X is the set $k[X]$ of all regular functions[2] on X with pointwise addition and multiplication:

$$k[X] = \{f|_X \mid f \in k[x_1, \ldots, x_n]\}$$

$$(f + g)(\mathbf{a}) = f(\mathbf{a}) + g(\mathbf{a}), (fg)(\mathbf{a}) = f(\mathbf{a})g(\mathbf{a}) \text{ for all } P \in X.$$

The set $k[X]$ with the operations defined above is a commutative ring.

There is a *restriction map* $\pi : k[x_1, \ldots, x_n] \to k[X]$ given by $\pi(f) = f|_X$. This is a ring homomorphism. We will be particularly interested in its kernel.

Definition 1.19. For any subset W of $\mathbb{A}^n_k$, let $I(W)$ denote the set of all polynomials that vanish at every point of W:

$$I(W) = \{f \in k[x_1, \ldots, x_n] \mid f(a_1, \ldots, a_n) = 0 \text{ for all } (a_1, \ldots, a_n) \in W\}.$$

[2] $k[X]$ is not to be confused with $k[x_1, \ldots, x_n]$. The former is a quotient ring of the latter.

Proposition 1.20. *Let $X \subseteq \mathbb{A}^n_k$ be an affine variety. Its ring of regular function is given (up to isomorphism) by the formula* [3]

$$k[X] \cong \frac{k[x_1, \ldots, x_n]}{I(X)}.$$

Proof. Consider the restriction homomorphism $\pi : k[x_1, \ldots, x_n] \to k[X]$ given by $\pi(f) = f|_X$. Then $I(X) = \text{Ker}(\pi)$ (in particular this set is an ideal) and the result follows by the First Isomorphism Theorem. □

Example 1.21. We have $k[\mathbb{A}^n] = k[x_1, \ldots, x_n]$ and $k[\text{one point}] = k$ since for $\mathbf{a} = (a_1, \ldots, a_n)$, we have that $I(\{\mathbf{a}\}) = (x_1 - a_1, \ldots, x_n - a_n)$ is a maximal ideal in $k[x_1, \ldots, x_n]$. The surjection $\pi : k[\mathbb{A}^n] \to k[\{\mathbf{a}\}], f \mapsto f(\mathbf{a})$ is evaluation at $\mathbf{a}$.

Example 1.22. Let $C = V(f) \subseteq \mathbb{A}^2_k$, where $f = y^2 - x^3$. This is called the *cuspidal cubic* curve.Then $I_C = (f)$ (this is not obvious - check!) and

$$k[C] \cong \frac{k[x, y]}{(y^2 - x^3)}.$$

Note that a regular function on C can be exhibited in a way that doesn't make it obvious that it is regular, for example,

$$g(x, y) = \begin{cases} \frac{y^2}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is regular because on C $y^2 = x^3$ and so $g(x, y) = x^2|_C$.

Monday, January 27

Let's look at some properties that the set $I(W)$ from Definition 1.19 has.

Proposition 1.23. *1. For any subset W of $\mathbb{A}^n_k$, $I(W)$ is a radical ideal in $k[x_1, \ldots, x_n]$.*

2. If W and W' are two subsets of $\mathbb{A}^n_k$, then:

(a) If $W \subseteq W'$ then $I(W) \supseteq I(W')$.

(b) $I(W \cup W') = I(W) \cap I(W')$.

Proof. (1) If $g \in I(W)$ and $f \in k[x_1, \ldots, x_n]$ then for all $P \in I(W)$ we have $(fg)(P) = f(P)g(P) = f(P)0 = 0$ and hence $fg \in I(W)$. Similarly one shows $I(W)$ is closed under addition. Since clearly $0 \in I(W)$, $I(W)$ is an ideal. If $f^n \in I(W)$ for some $n \geq 1$, then for all $P \in W$ we have $0 = (f^n)(P) = (f(P))^n$ and hence $f(P) = 0$. So $f \in I(W)$, and this proves $I(W)$ is radical.

Part (2) follows directly from the definitions. □

[3] We will often think of the two rings in this formula as being identical, writing $=$ instead of $\cong$.

Exercise 1.24. Prove that $I(\mathbb{A}^n) = (0)$ if and only if k is infinite.

Corollary 1.25. *If X is an affine variety, the coordinate ring $k[X]$ is a reduced k-algebra, that is, zero is the only nilpotent element.*

Proof. By Proposition 1.20 we have $k[X] = k[x_1, \dots, x_n]/I(X)$ and by Proposition 1.23 $I(X)$ is radical. Then $f^t = 0$ in $k[X]$ iff $f^t \in I(X)$ iff $f \in I(X)$ iff $f = 0$ in $k[X]$. □

1.2.2 Regular maps

What should be a morphism $X \to Y$ where $X \subseteq \mathbb{A}^n_k$ and $Y \subseteq \mathbb{A}^m_k$ are varieties? As before we would like the inclusion $Y \hookrightarrow \mathbb{A}^m_k$ to be a morphism and also the projection (say, on the first coordinate) $\mathbb{A}^m_k \twoheadrightarrow \mathbb{A}^1_k$ to be a morphism. If we compose these maps

$$X \to Y \hookrightarrow \mathbb{A}^m_k \twoheadrightarrow \mathbb{A}^1_k$$

we get a morphism $X \to \mathbb{A}^1_k$ which we know should be the restriction of a polynomial function $f_1 \in k[x_1, \dots, x_n]$. So the first coordinate of the map $X \to Y$ is given by $f_1|_X$ and, by the same argument, for each i the i-th coordinate is given by $f_i|_X$ for some $f_i \in k[x_1, \dots, x_n]$. Thus

$$f(\mathbf{a}) = (f_1|_X(\mathbf{a}), \dots, f_m|_X(\mathbf{a})) \text{ for all } \mathbf{a} \in X.$$

Definition 1.26. If $X \subseteq \mathbb{A}^n_k$ and $Y \subseteq \mathbb{A}^m_k$ are varieties, a morphism or *regular map* $X \to Y$ is a tuple of polynomial functions $f_i \in k[x_1, \dots, x_n]$ restricted to X

$$f = (f_1|_X, \dots, f_m|_X).$$

Exercise 1.27. Let $G_m = V(xy - 1) \subseteq \mathbb{A}^2$. Show the following are regular maps

- componentwise multiplication $G_m \times G_m \to G_m, ((a_1, a_2), (b_1, b_2)) \mapsto (a_1 b_1, a_2 b_2)$
- componentwise inverse $G_m \to G_m, (a_1, a_2) \mapsto (a_1^{-1}, a_2^{-1})$.

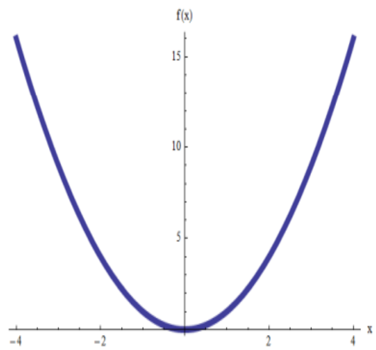
Show that with these operations G_m is isomorphic as a group with $(k, \cdot)$.

Exercise 1.28.

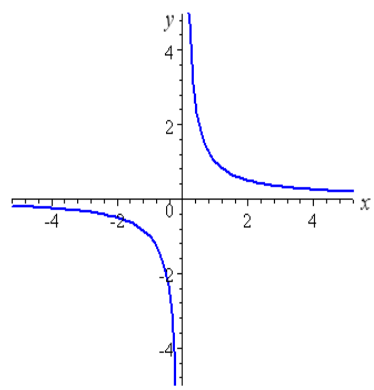
- Prove that regular maps $X \to \mathbb{A}^1_K$ are the regular functions.
- Prove that the inclusion $X \hookrightarrow \mathbb{A}^n_k$ is a regular map.
- Prove that compositions of regular maps are regular maps.
- Prove that a regular map $f : X \to \mathbb{A}^m_k$ has image in $Y \subseteq \mathbb{A}^m_k$, so is really a map $f : X \to Y$ if and only if $g \circ f = 0$ in $k[X]$ for all $g \in I(Y)$.

Definition 1.29. Two affine varieties X and Y are *isomorphic* if there are regular maps $g : X \to Y$ and $h : Y \to X$ such that $g \circ h = \mathrm{id}_Y$ and $h \circ g = \mathrm{id}_X$, in which case each of g an h is referred to as an *isomorphism* of affine varieties.

Example 1.30. Let $Z = V(y - x^2)$ be the parabola (graph of $f(x) = x^2$). Then Z is isomorphic to $\mathbb{A}^1_k$ via the mutually inverse morphisms $g : Z \to \mathbb{A}^1_K, g(x, y) = x$ and $h : \mathbb{A}^1_k \to Z, h(x) = (x, x^2)$.

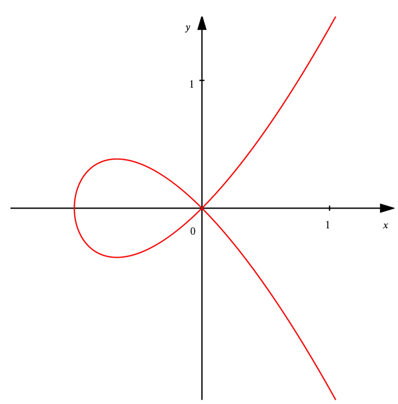


Example 1.31. Let $X = V(xy-1) \subseteq \mathbb{A}_k^2$ (i.e., X is a hyperbola) and define $g : X \to \mathbb{A}_K^1$ by $g(a,b) = a$. Then g is a regular map (indeed, it's given by a linear polynomial) and its image is $\mathbb{A}_K^1 \setminus \{0\}$, which is *not* an affine variety of $\mathbb{A}_k^1$. So, **the set-theoretic image of a regular map need not be a variety**.



Example 1.32. Let X be the nodal cubic in $\mathbb{A}_k^2$ given by

$$X = V(y^2 - x^2(x+1)).$$



Define a map

$$g : \mathbb{A}_K^1 \to X \quad g(t) = (t^2 - 1, t^3 - t).$$

g is a regular map since it is represented by polynomials functions and for any $t \in \mathbb{A}^1_K$, we have $g(t) \in X$ since

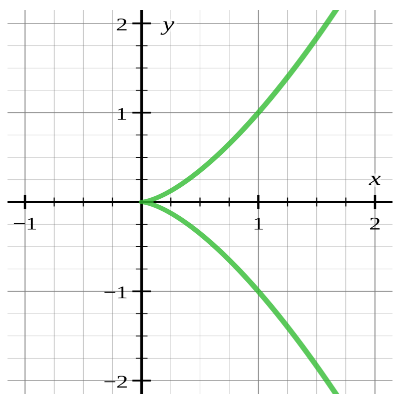
$$(t^3 - t)^2 - (t^2 - 1)^2(t^2 - 1 + 1) = t^6 - 2t^4 + t^2 - (t^4 - 2t^2 + 1)t^2 = 0.$$

The function g is surjective and the preimage of every point on X is single point with one exception: the fiber of $(0, 0) \in Z$ consists of two points, 1 and -1, provided $\mathrm{char}(K) \neq 2$. Since g is not a set-theoretic bijection it cannot be an isomorphism.

Remark 1.33. An isomorphism of algebraic sets must be a bijection, but the converse is not true according to the example below.

Example 1.34. Let C be the *cuspidal cubic curve*:

$$Y = V(y^2 - x^3) \subseteq \mathbb{A}^2_K.$$



Define

$$g : \mathbb{A}^1_K \to Y \quad g(t) = (t^2, t^3).$$

g is a regular map from $\mathbb{A}^1_K$ to C since the component functions are polynomial functions of t and $(t^3)^2 - (t^2)^3 = 0$ for all t. Note that g **is a bijection, but, it is *not* an isomorphism of affine varieties**! We will justify this later.

1.2.3 Pullbacks

In this section we see how regular maps interact with regular functions, i.e., how they induce maps on coordinate rings.

Definition 1.35. Suppose $f : X \to Y$ is a regular map and $u : Y \to \mathbb{A}^1_k$ a regular function. Define the *pullback* of u by f as

$$f^*(u) : X \to A^1_k, \quad f^*(u) = u \circ f.$$

Concretely, if $f = (f_1|_X, \ldots, f_m|_X)$, then $f^*(u(y_1, \ldots, y_m)) = u(f_1, \ldots, f_m)$.

The pullback construction induces a ring homomorphism

Proposition 1.36. *Suppose $f : X \to Y$ is a regular map and $u : Y \to \mathbb{A}^1_k$ a regular function. Then the pullback by f induces a k-algebra homomorphism*

$$f^* : k[Y] \to k[X], \quad f^*(u) = u \circ f.$$

Proof. First we show f^* is well-defined. Indeed, $f^*(u) = u \circ f$ is a regular function on Y as compositions of polynomials are polynomials.

Recall that a k-algebra homomorphism is a ring homomorphism that is also k-linear. The properties that make this map such a homomorphism are easily checked:

$$\begin{aligned} f^*(\mathrm{id}_{k[Y]}) &= \mathrm{id}_{k[Y]} \circ f = \mathrm{id}_{k[X]} \\ f^*(u+v) &= (u+v) \circ f = u \circ f + v \circ f = f^*(u) + f^*(v) \\ f^*(uv) &= (uv) \circ f = (u \circ f)(v \circ f) = f^*(u) f^*(v) \\ f^*(cu) &= (cu) \circ f = c \cdot u \circ f = c f^*(u) \text{ for } c \in k. \end{aligned}$$

□

Exercise 1.37 (Properties of pullbacks)**.** Prove that if X, Y, Z are affine varieties and $f : Y \to Z, g : X \to Y$ are regular maps then

- $(\mathrm{id}_X)^* = \mathrm{id}_{k[X]}$
- $(f \circ g)^* = g^* \circ f^*$.

Exercise 1.38. Prove that if X, Y are affine varieties and $h : k[Y] \to k[X]$ is a k-algebra map then there exists a unique regular map $f : X \to Y$ such that $h = f^*$.

Exercise 1.39. Consider the regular maps $f, g : \mathbb{A}^1_k \to \mathbb{A}^2_k, f(t) = (t, t^2), g(t) = (t^2, t^3)$. Let X and Y denote their respective images and view $f : \mathbb{A}^1_k \to X, g : \mathbb{A}^1_k \to Y$. What are the pullbacks of these maps? Are f, g injective/surjective/bijective? Are the pulback maps injective/surjective/bijective?

Wednesday, January 29

1.3 The algebra-geometry correspondence

1.3.1 Nullstellensatz

Henceforth, we assume k is algebraically closed unless otherwise specified.

There are several versions of the Nullstellensatz. Proofs can be found in Math 905.

Theorem 1.40 (Algebraic Nullstellensatz)**.** *Let k and E be fields. If E is a finitely generated k-algebra then E is a finite algebraic extension of k. In particular, if k is algebraically closed, then $E = k$.*

Theorem 1.41 (Weak Nullstellensatz - I)**.** *If I is an ideal in $k[x_1, \dots, x_n]$ and $V(I) = \emptyset$ then $I = (1) = k[x_1, \dots, x_n]$.*

Theorem 1.42 (Weak Nullstellensatz - II). *If $\mathfrak{m}$ is a maximal ideal in $k[x_1,\dots,x_n]$ then there exists a unique $\mathbf{a} = (a_1,\dots,a_n) \in \mathbb{A}^n_k$ so that $\mathfrak{m} = (x_1 - a_1,\dots,x_n - a_n)$.*

Theorem 1.43 (Strong Nullstellensatz). [4] *Assume k is algebraically closed. Then $V(-)$ and $I(-)$ are order-reversing mutually inverse bijections between the partially ordered sets (under containment)*

$$\textit{Affine Varieties in } \mathbb{A}^n_k \overset{V}{\underset{I}{\longleftrightarrow}} \textit{Radical Ideals in } k[x_1,\dots,x_n].$$

Specifically,

$$I(V(J)) = J \text{ and } V(I(X)) = X$$

for any radical ideal J of $k[x_1,\dots,x_n]$ and any affine variety X in $\mathbb{A}^n_k$.

Exercise 1.44. Prove the second part of the Strong Nullstellensatz, assuming the first part is true. That is, assume that $I(V(J)) = J$ for any radical ideal J of $k[x_1,\dots,x_n]$ and prove that $V(I(X)) = X$ for any affine variety X in $\mathbb{A}^n_k$.

There are slightly following more general statements of the Strong Nullstellensatz:

Corollary 1.45. *If k is algebraically closed, then for any set S of polynomials we have $I(V(S)) = \sqrt{(S)}$. In particular, for any ideal J we have*

$$I(V(J)) = \sqrt{J}.$$

Moreover, if W is any subset of $\mathbb{A}^n_k$, $V(I(W))$ is the unique smallest affine variety in $\mathbb{A}^n_k$ that contains W, which we will denote by $\overline{W}$ and call the Zariski closure *of W. Using this notation, we have*

$$V(I(W)) = \overline{W}.$$

Exercise 1.46. Prove Corollary 1.45.

The hypothesis that k is algebraically closed is crucial.

Example 1.47. For $J = (x^2 + y^2)$ and $k = \mathbb{R}$, we have that J is a radical ideal and yet $I(V(J)) = (x, y) \supsetneq J$.

More concretely, the two systems of equations

$$x^2 + y^2 = 0 \qquad \text{and} \qquad \begin{cases} x &= 0 \\ y &= 0 \end{cases}$$

have the same set of solutions (over $\mathbb{R}$), and yet $x = 0$ is *not* an algebraic consequence of $x^2 + y^2 = 0$ (no power of x is a $\mathbb{R}[x, y]$-multiple of $x^2 + y^2$).

[4] This theorem is phrased a little differently from what was called the Strong Nullstellensatz in Math 905. The first assertion is (a particular case of) what was called the Strong Nullstellensatz in Math 905. However this is the "hard" part of the theorem. See Exercise 1.44. What was called the Strong Nullstellensatz in Math 905 is a consequence of the first assertion of Theorem 1.43; see Exercise 1.45.

Let's give some applications of the Strong Nulistellenstaz:

Corollary 1.48. *Suppose k is algebraically closed.*

1. *If X and X' are two affine varieties in $\mathbb{A}^n_k$, then $I(X \cap X') = \sqrt{I(X) + I(X')}$.*
2. *If J and J' are two radical ideals in $k[x_1, \ldots, x_n]$ then $V(J \cap J') = V(J) \cup V(J')$.*
3. *If X and Y are disjoint affine varieties then $k[X \cup Y] \cong k[X] \oplus k[Y]$.*

Proof. 1. Since both sides are radical ideals, and $V(-)$ is a bijection, by the Nullstellensatz it suffices to prove

$$V(I(X \cap X')) = V(\sqrt{I(X) + I(X')}).$$

This holds since $V(I(X \cap X')) = X \cap X'$ by Nullstellensatz and using Exercise 1.15

$$V(\sqrt{I(X) + I(X')}) = V(I(X) + I(X')) = V(I(X)) \cap V(I(X')) = X \cap X'.$$

2. We apply $I(-)$ to both sides: $I(V(J \cap J')) = J \cap J'$ by the Nullstellensatz while

$$I(V(J) \cup V(J')) = I(V(J)) \cap I(V(J')) = J \cap J'.$$

Because $I(-)$ is a bijection, we conclude the desired equality.

3. This is a homework problem. □

Example 1.49. Show that Corollary 1.48 3. is false without the disjoint hypothesis.

Example 1.50. The Corollary is false over $\mathbb{R}$: Take $W = V(y - x^2)$ and $W' = V(y + 1)$ in $\mathbb{A}^2_{\mathbb{R}}$. Then $I(W) = (y - x^2)$ and $I(W') = (y + 1)$ (why?) and hence $\sqrt{I(W) + I(W')} \subsetneq k[x, y]$ (since $k[x, y]/(y - x^2, y + 1) \cong k[x]/(x^2 + 1) \neq 0$). But $W \cap W' = \emptyset$ and so $I(W \cap W') = k[x, y]$.

1.3.2 An equivalence of categories

In mathematics, a *category* is a collection of "objects" that are linked by "arrows" (morphisms). A category has two basic properties: the ability to compose the arrows associatively and the existence of an identity arrow for each object. A simple example is the category of sets, whose objects are sets and whose arrows are functions. Other examples include: the category of groups, whose objects are groups and whose arrows are group homomorphisms, the category of rings, whose objects are rings and whose arrows are ring homomorphisms, and many others.

We won't prove the following but it is true.

Theorem 1.51. *The collections of affine varieties over a fixed field k and the regular between them form a category $\langle\langle$Affine Varieties$\rangle\rangle_k$.*

A *contravariant functor* F is a map between two categories $F : \mathcal{C} \to \mathcal{D}$ such that

1. it associates to each object $X \in \mathcal{C}$ an object $F(X) \in \mathcal{D}$
2. it associates to each morphism $f : X \to Y$ in $\mathcal{C}$ a morphism $F(X) : F(Y) \to F(X)$ in $\mathcal{D}$
3. $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for every X in $\mathcal{C}$
4. $F(g \circ f) = F(f) \circ F(g)$ for all morphisms f, g in $\mathcal{C}$ so that $g \circ f$ makes sense..

An *equivalence of categories* is a relation between two categories that establishes that these categories are "essentially the same". A functor $F : \mathcal{C} \to \mathcal{D}$ yields an equivalence of categories if and only if it is simultaneously:

1. *full*: every morphism between objects of $\mathcal{D}$ is the image by F of some morphism between objects of $\mathcal{C}$;
2. *faithful*: the map induced by F on morphisms is injective;
3. *essentially surjective*, each object in $\mathcal{D}$ is isomorphic to an object of the form $F(c)$, for some $c \in C$.

Theorem 1.52. *There is a contravariant functor*

$$k[-] : \langle\langle \textit{Affine Varieties} \rangle\rangle_k \to \langle\langle \textit{Reduced, finitely generated } k\textit{-algebras} \rangle\rangle$$

which takes an affine variety X to its ring of regular functions $k[X]$ and takes a regular map $X \to Y$ to its pullback $f^* : k[Y] \to k[X]$.
This functor induces an equivalence of categories between the two categories listed above.

Proof. To see that $k[-]$ <u>is a functor</u> one has to check that

- $(\mathrm{id}_X)^* = \mathrm{id}_{k[X]}$
- $(g \circ f)^* = f^* \circ g^*$.

These follow from Definition 1.35. For example, associativity of composition gives

$$(g \circ f)^*(u) = u \circ (g \circ f) = (u \circ g) \circ f = f^*(u \circ g) = f^*(g^*(u)) = (f^* \circ g^*)(u).$$

To see that the functor is <u>full</u> we need to check: if $h : k[Y] \to k[X]$ is a k-algebra map then there exists a regular map $f : X \to Y$ such that $h = f^*$.

Given a k-algebra map $h : k[Y] \to k[X]$, that is

$$h : \frac{k[y_1, \dots y_m]}{I(Y)} \to \frac{k[x_1, \dots x_n]}{I(X)},$$

let h_i be any lift of $h(y_i) \in k[X]$ to $k[x_1, \dots x_n]$ and define a regular map

$$f : X \to \mathbb{A}_k^m, \quad f(\mathbf{a}) = (h_1|_X(\mathbf{a}), \dots, h_m|_X(\mathbf{a})).$$

We will show that the image of this map lands in Y. By a homework problem it suffices to show that for all $g \in I(Y)$ we have $g \circ f = 0$ in $k[X]$. This amounts to computing

$$g \circ f = g(h_1, h_2, \dots, h_n) = g(h(y_1), \dots, h(y_n)) = h(g(x))$$

since h is a ring homomorphism. But since $g \in I(Y)$, $g = 0$ in $k[Y]$ and thus $h(g) = 0$ in $k[X]$ as desired.

To see that the functor is faithful we need to check that the f we found above is uniquely determined by h. Let $f = (f_1, \ldots, f_m)$ with $f_i \in k[X]$. Since $h = f^*$ we have

$$f_i = f^*(y_i) = h(y_i).$$

This shows that f is unique.

To see that the functor is essentially surjective we need to check that every reduced, finitely generated k-algebra is isomorphic to $k[X]$ for some affine variety X. A finitely generated k-algebra can be described as $k[x_1, \ldots, x_n]/I$ for some ideal I. It is reduced if and only if I is a radical ideal. Set $X = V(I)$. Then by the Nullstellensatz $I(X) = I$ and by Proposition 1.20 we have $k[X] \cong k[x_1, \ldots, x_n]/I(X) = k[x_1, \ldots, x_n]/I$, as desired. □

In the proof above we have used the following important fact:

Lemma 1.53 (A formula for computing pullback)**.** *If $f : X \to Y$ is a regular map given by $f(\mathbf{a}) = (f_1(\mathbf{a}), \ldots, f_m(\mathbf{a}))$ then for any $u(y_1, \ldots, y_m) \in k[Y]$ we have*

$$f^*(u) = u(f_1, \ldots, f_m) = u(f^*(y_1), \ldots, f^*(y_m)).$$

Equivalences of categories take isomorphisms to isomorphisms. Thus we now see that the coordinate ring functor detects isomorphisms of affine varieties.

Proposition 1.54. *A regular map $f : X \to Y$ is an isomorphism of affine varieties if and only if $f^* : k[Y] \to k[X]$ is a ring isomorphism.*

Exercise 1.55. Prove the Proposition above using the properties listed in Exercise 1.37 and in Exercise 1.38 (which was solved in the proof of Theorem 1.52).

Since we know that every k-algebra homomorphism $h : k[Y] \to k[X]$ is given by $h = f^*$ for some regular map $f : X \to Y$ (see the proof of Theorem 1.52) we obtain.

Theorem 1.56. *Two affine varieties are isomorphic if and only if their coordinate rings are isomorphic.*

Exercise 1.57. Prove that the parabola $X = V(y - x^2)$ is isomorphic to $\mathbb{A}^1_k$.

Exercise 1.58. Prove that the cuspidal cubic $Y = V(y^2 - x^3)$ is not isomorphic to $\mathbb{A}^1_k$.

Friday, January 31

1.4 Zariski topology

1.4.1 The Zariski topology

Recall that a *topology* on a set X is a specified collection of subsets of X, known as the *open subsets*, such that

1. X is an open subset of X,
2. $\emptyset$ is an open subset of X,
3. the intersection of any two open subsets it open, and
4. an arbitrary union of open subsets is open.

Such a specified collection of open subsets makes X into a *topological space*.

In any topological space, a *closed subset* is any subset that is the complement of an open one. One may equivalently describe a topology as a collection of specified closed subsets such that

1. X is a closed subset of X,
2. $\emptyset$ is a closed subset of X,
3. the intersection of an arbitrary collection of closed subsets is closed, and
4. the union of two closed subsets is closed.

By what we have proved in Proposition 1.14 about affine varieties, we may define:

Definition 1.59. The *Zariski topology* on $\mathbb{A}_k^n$ is the topology in which a subset X of $\mathbb{A}_k^n$ is *closed* if and only if it is an affine variety (i.e., $X = V(I)$ for some ideal I of the polynomial ring) and a subset U of $\mathbb{A}_k^n$ is *open* iff $U = \mathbb{A}_k^n \setminus V(I)$ for some ideal I.

Example 1.60. In $\mathbb{A}_k^1$ the only proper closed subsets of are the finite sets (including the empty set).

Example 1.61. In $\mathbb{A}_k^2$ the proper closed subsets are finite unions of curves (hypersurfaces) and points in $\mathbb{A}_k^2$.

Exercise 1.62. Show that the Zariski topology on $\mathbb{A}^{m+n} = \mathbb{A}^m \times \mathbb{A}^n$ is **not** the same as the product topology of the Zariski topologies of the two factors.

We can also talk about the Zariski topology on an affine variety.

Definition 1.63. Given a variety X in $\mathbb{A}_k^n$, the *Zariski topology* on X is the subspace topology, that is, a subset of X is open in the subspace topology if and only if it is the intersection of X with an open set of $\mathbb{A}^n$ and, equivalently, a subset of X is closed in the subspace topology if and only if it is the intersection of X with a closed set of $\mathbb{A}^n$.

Remark 1.64.

- A subset U of an affine variety X is open iff it has the form $U = X \setminus V(I)$ for some ideal I in $k[x_1, \ldots, x_n]$.
- A subset C of an affine variety X is closed iff C is an affine subvariety of X (i.e., an affine variety in $\mathbb{A}_k^n$ that is contained in X).

1.4.2 Irreducible varieties and irreducible decomposition

Definition 1.65. An affine variety is *irreducible* if it cannot be written non-trivially as a union of affine variaties — that is, X is irreducible provided whenever $X = Y \cup Z$ with Y and Z affine varieties, we have $Y = X$ or $Z = X$.

Definition 1.66. An affine variety is *connected* if it cannot be written non-trivially as a disjoint union of affine varieties — that is, X is connected provided whenever $X = Y \cup Z$ with Y and Z affine varieties such that $Y \cap Z = \emptyset$, we have $Y = X$ or $Z = X$.

Irreducible implies connected but not the other way around.

Example 1.67.
1. $V(xy) = V(x) \cup V(y)$ is connected but reducible (the union of the x and y axes);
2. $V(xy, x^2 - x) = V(x) \cup V(x-1, y)$ is disconnected and reducible (the union of the y axis and the point $(1, 0)$);
3. $V(x - y^2)$ is connected and irreducible; see Exercise 1.68.
4. $V(xy, xz) = V(x) \cup V(y, z)$ in $\mathbb{A}^3_k$ is connected and reducible as it is the union of a line and a plane, specifically the x-axis and the yz-plane.

Exercise 1.68. Prove that a hypersurface $X = V(f(x_1, \dots, x_n))$ in $\mathbb{A}^n_k$ is irreducible if and only if f is an irreducible polynomial or a power of an irreducible polynomial. *Hint:* First reduce to the case that $f \in k[x_1, \dots, x_n]$ is a polynomial without repeated factors and then prove $X = V(f)$ is irreducible if and only if f is irreducible.

Exercise 1.69. Show that the following are equivalent for an affine variety X
1. X is irreducible
2. every non-empty open subset U of X is dense in X i.e. $\overline{U} = X$
3. the intersection of any two non-empty open subset of X is non-empty.

Proposition 1.70 (Irreducibility criterion)**.** *The following are equivalent for an affine variety X*
1. *X is irreducible*
2. *$I(X)$ is a prime ideal*
3. *$k[X]$ is an integral domain.*

More generally, the collection of irreducible affine subvarieties of a given affine variety is in bijective correspondence with the collection of primes ideals in its coordinate ring.

Proof. The fact that 2. and 3. are equivalent follows from Math 818 using Proposition 1.20. Thus we will only prove that 1. and 2. are equivalent below.

If X is not irreducible, then $X = Y \cup Z$ for proper closed subsets Y and Z of X. It follows that $I(X) \subsetneq I(Y)$, $I(X) \subsetneq I(Z)$, and $I(X) = I(Y) \cap I(Z)$. So pick $f \in I(Y) \setminus I(X)$ and $g \in I(Z) \setminus I(X)$. Then $fg \in I(X)$, proving $I(X)$ is not prime.

Suppose $I(X)$ is not prime. Then there are polynomials f and g neither of which is in $I(X)$ and yet $fg \in I(X)$. Set $J = I(X) + (f)$ and $L = I(X) + (g)$ and $Y = V(J)$ and

$Z = V(L)$. Then $Y \subsetneq X$, $Z \subsetneq X$ and $Y \cup Z = V(J) \cup V(L) = V(J \cdot L) = V(I(X)) = X$ since $I \cdot L = I(X)$.

For the second assertion, we already know there is a bijective correspondence between affine subvarieties of an affine variety V and radical ideals of $A(V)$, given by $W \mapsto I(W)/I(V)$. The assertion holds since $I(W)/I(V)$ is prime in $A(V)$ if and only if $I(W)$ is prime in $k[x_1, \ldots, x_n]$. □

If a variety is reducible, we can decompose it as a finite union of irreducible varieties. To prove this we need a preliminary lemma.

Definition 1.71. A topological space is *noetherian* if it satisfies the descending chain condition on closed sets: any descending chain $X_1 \supsetneq X_2 \supsetneq X_3 \supsetneq \ldots$ of closed subsets must eventually stabilize.

Lemma 1.72. *If k is algebraically closed, then $\mathbb{A}^n_k$ with its Zariski topology is noetherian. More generally, any affine variety X of $\mathbb{A}^n_k$ with its Zariski topology is noetherian.*

Proof. A descending chain $X_1 \supsetneq X_2 \supsetneq X_3 \supsetneq \ldots$ of closed subsets of X corresponds to an ascending chain $I(X_1) \subsetneq I(X_2) \subsetneq I(X_3) \subsetneq \ldots$ of ideals of $k[x_1, \ldots, x_n]$, where we have used the Nullstellensatz (specifically fact that the function $I(-)$ is a bijection) to conclude the non-equalities in the chain of ideals. Since the polynomial ring is noetherian, the chain of ideals must stabilize so the chain of varieties stabilizes too (again using the Nullstellensatz). □

Theorem 1.73. *Every affine variety X is a finite union of irreducible closed subsets, $X = X_1 \cup \cdots \cup X_m$, such that $X_i \not\subseteq X_j$ for all $i \neq j$. Moreover, the list of such subsets is unique up to ordering, and are known as the* irreducible components *of X.*

Proof. In general, if X is any noetherian topological space, then the conclusion of the theorem holds. The existence is given as follows: If X is irreducible, then we may take $m = 1$ and $X_1 = X$. If not, then $X = X_1 \cup Y_1$ for two proper closed subsets X_1 and Y_1 of X. If both X_1 and Y_1 are irreducible, we are done; otherwise we can decompose at least one of them — without loss say it is Y_1 that decomposes. Then $Y_1 = X_2 \cup Y_2$ with X_2 and Y_2 closed a proper in Y_1. Continuing in the fashion gives a strictly descending chain $X \supsetneq Y_1 \supsetneq Y_2 \supsetneq$ of closed subsets. By noetherianity, this process must stop after a finite number of steps, and thus $X = X_1 \cup \cdots \cup X_m$ for some irreducible closed subsets $X_1, \ldots, X_m$. If $X_i \subseteq X_j$ for some $i \neq j$, delete X_i from the list.

Now for the uniqueness: suppose

$$X = X_1 \cup \cdots \cup X_m = Y_1 \cup \cdots \cup Y_n$$

such that X_i, Y_j are irreducible varieties, $X_i \not\subseteq X_j$ and $Y_i \not\subseteq Y_j$ for any i, j. Then

$$\begin{aligned} X_i &= (X_i \cap Y_1) \cup \cdots \cup (X_i \cap Y_n) \\ Y_j &= (X_1 \cap Y_j) \cup \cdots \cup (X_m \cap Y_j), \end{aligned}$$

but since X_i, Y_j are irreducible and $X_i \cap Y_j$ are affine varieties, it must be that $X_i \cap Y_j = X_i = Y_j$ whenever $X_i \cap Y_j \neq \emptyset$. The desired conclusion regarding uniqueness follows. □

How do we find the irreducible components? The answer is given by primary decomposition. Recall from Math 905:

Theorem 1.74. *Any ideal in a noetherian ring decomposes as an intersection of primary ideals. In particular any radical ideal J in a noetherian ring decomposes as*

$$J = P_1 \cap P_2 \cap \cdots \cap P_n$$

where $P_1, \ldots, P_n$ are prime ideals such that $P_i \not\subseteq P_j$ for all $i \neq j$ and this decomposition is unique up to re-ordering the prime ideals. (The primes $P_1, \ldots, P_n$ above are called the associated primes *of J.)*

As a corollary we have the following correspondence between associated primes and irreducible components:

Remark 1.75. If J is a radical ideal and if $J = P_1 \cap P_2 \cap \cdots \cap P_n$ with P_i prime ideals as above then by Proposition 1.70 $V(P_i)$ is an irreducible affine variety for each i and by Corollary 1.48 (2.) we have

$$X = V(J) = V(P_1 \cap P_2 \cap \cdots \cap P_n) = V(P_1) \cup V(P_2) \cup \cdots \cup V(P_n)$$

is the irreducible decomposition of X.

Monday, February 3

1.4.3 Distinguished open sets and compactness

One particular kind of Zariski open subset is particularly important:

Definition 1.76. A *distinguished open subset* of an affine variety X is a set of the form

$$D_X(f) = \{P \in X \mid f(P) \neq 0\}$$

for some element $f \in k[X]$. We can also think of $D_X(f) = X \cap D_{\mathbb{A}^n_k}(f)$ for $f \in k[x_1, \ldots, x_n]$.)

Exercise 1.77. Let X be any affine variety. Show $D_X(f) \cap D_X(g) = D_X(fg)$; in particular, the intersection of finitely many distinguished open subsets of X is again distinguished.

Recall that a collection of subsets $\mathcal{B}$ of a set T is a *basis for a topology* on T if:

1. for every point P there is a $U \in \mathcal{B}$ with $P \in U$ and
2. if $P \in U_1 \cap U_2$ for some $U_1, U_2 \in \mathcal{B}$, there is $U_3 \in \mathcal{B}$ such that $P \in U_3 \subseteq U_1 \cap U_2$.

The topology generated by such a basis is the collection of subsets of T that are arbitrary unions of members of $\mathcal{B}$.

Exercise 1.78. Let X be any affine variety. Show that the distinguished open sets of X form a basis for the Zarsiki topology. So, you are being asked to show the above two properties hold for the collection of distinguished open sets and that every Zariski open set is a union of distinguished ones.

An important property of the Zariski topology is that it is compact. This will allow us to reduce proofs involving possibly infinitley many sets whose union covers X to finitely many.

Definition 1.79. A topological space X is *(quasi-)compact* if every open cover of X by open subsets admits a finite subcover.

Proposition 1.80 (Quasi-compactness of the Zariski topology)**.** *Every affine variety X is (quasi-)compact for the Zariski topology. In fact, every open subset of X is (quasi-)compact.*

Proof. (Quasi-)compactness is equivalent to a statement on closed sets: if $\{C_\alpha\}$ is a collection of closed subsets of X such that $\bigcap_\alpha C_\alpha = \emptyset$ then $C_{\alpha_1} \cap \cdots \cap C_{\alpha_m} = \emptyset$ for some finite list $\alpha_1, \dots, \alpha_m$ of indices.

We may translate the first assertion into a statement about ideals in the ring $R = k[X]$. We need to prove that if $\{I_\alpha\}$ is a collection of ideals such that $\sum_\alpha I_\alpha = R$ then $I_{\alpha_1} + \cdots + I_{\alpha_m} = R$ for some finite sub-collection. This holds since $\sum_\alpha I_\alpha = R$ iff $1 \in \sum_\alpha I_\alpha$ iff $1 \in I_{\alpha_1} + \cdots + I_{\alpha_m}$ for some $\alpha_1, \dots, \alpha_m$.

I'll just sketch the proof of the second assertion: Let U be an open subset of an affine variety X, say $U = X \setminus V(J)$ for an ideal J. We have $J = (g_1, \dots, g_m)$ by the Hilbert Basis Theorem, and so $U = \bigcup_{i=1}^m X \setminus V(g_i)$. Since this is a finite union, it suffices to show each $X \setminus V(g_i)$ is quasi-compact. So, without loss, we may assume $U = X \setminus V(g)$ for some g. The points of U are in bijective correspondence with the maximal ideals of the ring $R[1/g]$ and the closed subsets are given by ideals of this ring. So, the same proof as in the first part applies. □

In the previous proof sketch we have seen that the ring $k[X][1/g]$ acts like a kind of coordinate ring for $D_X(g)$. We will explain this more rigorously in the next section.

We use the terminology quasi-compact to indicate that a space is compact but not Hausdorff.

Exercise 1.81. Show that the Zariski topology on $\mathbb{A}^n_k$ is not Hausdorff (provided $n > 0$). Indeed, show the intersection of any two non-empty open sets is non-empty.

1.5 The sheaf of regular functions

Example 1.82 (Projection from a point)**.** Consider the map that projects from the origin points in $\mathbb{A}^2_k$ to the line L with equation $y = 1$. Spcifically, if $(x, y) \in \mathbb{A}^2_k$ the line connecting the origin and this point intersects L at $(\frac{x}{y}, 1)$. Identifying $L \cong \mathbb{A}^1_k = k$ this gives a map $\varphi : \mathbb{A}^2_k \dashrightarrow \mathbb{A}^1_k = k, (x, y) \mapsto \frac{x}{y}$. Note that this map is *not* defined on all of $\mathbb{A}^2_k$. Indeed it is only defined on the open set $U = \{(x, y) : y \neq 0\}$. This if reflected by the fact that we represent ϕ by a dashed arrow. Moreover this map is not a regular function, but rather a rational function, that is, given by a quotient of two polynomials.

This shows that we will need to consider rational functions, i.e. quotients of polynomials and that these functions may only be defined on an open set. A natural class of functions to consider is that of *rational functions* of the form $\frac{g}{h}$ with $g, h \in k[X]$, which are defined on $D_X(h)$. If X is irreducible, so $k[X]$ is a domain, these are the elements of the fraction field

$$\mathrm{Frac}(k[X]) = \left\{ \frac{g}{h} \mid g, h \in k[X], h \neq 0 \right\}.$$

However for additional flexibility we'll consider all functions that agree with a rational function in some open neighborhood of each input. A way to think about these functions that we'll develop below is that they are obtained by "glueing" together functions of the form $\frac{g_i}{h_i}$ that agree on the overlaps of their domains $D_X(h_i) \cap D_X(h_j)$.

Definition 1.83. Let X be an affine variety in $\mathbb{A}^n_k$ and $U \subseteq X$ any open subset (for the Zariski topology).

A k-valued function $f : U \to k$ is *regular at a point* $\mathbf{a} \in U$ if there exist $g, h \in k[X]$ such that $h(\mathbf{a}) \neq 0$ and f agrees with $\frac{g}{h}$ in some neighborhood of $\mathbf{a}$. More formally: there is a Zariski open neighborhood U' of $\mathbf{a}$ in U, so that $\mathbf{a} \in U' \subseteq U$ and $f(P) = \frac{g(P)}{h(P)}$ for all $P \in U'$.

The function f is *regular on* U if it is regular at every point of U. The set of all regular functions on U is denoted $\mathcal{O}_X(U)$.

The set of all regular functions at $\mathbf{a}$ is the set of equivalence classes of regular functions at $\mathbf{a}$, where for Zariski open sets U_1, U_2 containing $\mathbf{a}$ and regular functions $f_1 : U_1 \to k$ and $f_2 : U_2 \to k$ we say they are equivalent if there is a Zariski open set $U_3 \subseteq U_1 \cap U_2$ so that $f_1|_{U_3} = f_2|_{U_3}$.

The set of all regular functions at $\mathbf{a}$ is denoted $\mathcal{O}_{X,\mathbf{a}}$.

Note that $\mathcal{O}_X(U), \mathcal{O}_{X,\mathbf{a}}$ are commutative rings with pointwise addition and multiplication.

Example 1.84. The map φ in Example 1.82 is regular at every point $\mathbf{a} = (a_1, a_2)$ with $a_2 \neq 0$ so $\varphi \in \mathcal{O}_{\mathbb{A}^2_k}(D(y))$.

Example 1.85. Every element of $k[X]$ determines a regular function on every open subset U of X: Given $f \in k[X]$, let $\alpha : U \to k$ be defined by $\alpha(P) = f(P)$, then in the definition of regular function we can take $U' = U$ for all $\mathbf{a}$.

Thus there is a map (in fact a k-algebra homomorphism)

$$\iota : k[X] \to \mathcal{O}_X(U), \quad f \mapsto f.$$

We will prove below in Theorem 1.89 that the map ι is in fact an isomorphism when $U = X$ — that is, there are no "interesting" functions that is regular at every point of an affine variety. In other words, the next theorem shows that the regular functions on X viewed as an open set of itself are the same as what we used to call regular functions, that is, the elements of $k[X]$.

First we need a lemma.

Lemma 1.86. *For an open subset U of an affine variety $X \subseteq \mathbb{A}^n_k$, a function $\alpha : U \to k$ is regular if and only if there is a finite cover $U = U_1 \cup \cdots \cup U_m$ of U by distinguished open subsets $U_i = D_X(h_i)$, for some $h_i \in k[X]$, and elements $f_1, \ldots, f_m \in k[X]$ such that, for each i, the function $\alpha|_{U_i} : U_i \to k$ is given by $\left(P \mapsto \frac{f_i(P)}{h_i(P)}\right)$ and f_i is identically zero on $X \setminus U_i$.*

Proof. The direction $\Leftarrow$ is clear from the definition.

For the other direction, say $\alpha : U \to k$ is a regular function. By definition we can write $U = \bigcup_i U_i$ for some (possibly infinite) collection of open subsets $\{U_i\}$ of U such that for each i we have $\alpha|_{U_i} = \frac{f_i}{g_i}|_{U_i}$ for some $f_i, g_i \in k[X]$ with g_i nonzero on U_i. Since the distinguished open subsets form a basis for the Zariski topology, by shrinking a bit we may assume each U_i is distinguished: $U_i = D_X(h_i)$ for $h_i \in k[X]$. By the quasi-compactness of U, we can take the collection $\{U_i\}$ to be finite. It remains to show that we may assume $g_i = h_i$ and that $V_X(f_i) \supseteq V_X(h_i)$.

Since $\frac{f_i}{g_i} = \frac{f_i h_i}{g_i h_i}$ conincide as functions on U_i, by replacing f_i with $f_i h_i$ and g_i with $g_i h_i$, we may assume $f_i(Q) = 0 = g_i(Q)$ for all $Q \notin U_i$. Since $g_i(Q) \neq 0$ for all $Q \in U_i$, this gives that $U_X(g_i) = U_X(h_i)$. By replacing h_i with g_i, in the notation, we may as well assume $g_i = h_i$. □

Theorem 1.87. *Let X be an affine variety in $\mathbb{A}^n_k$. The canonical map*

$$\iota : k[X] \to \mathcal{O}_X(X), \quad f \mapsto f.$$

is an isomorphism of k-algebras.

Proof. It is clear that the function ι is a homomorphism.

If f is in the kernel of ι, then $f \in I(X)$ so that $f = 0$ in $k[X]$. Thus ι is injective.

Now let φ be a regular function in the sense of Definition 1.83. By Lemma 1.86 there is a finite cover $U = U_1 \cup \cdots \cup U_m$ of X by distinguished open subsets $U_i = D_X(h_i)$, for some $h_i \in k[X]$, and elements $f_1, \ldots, f_m \in k[X]$ such that, for each i, the function $\alpha|_{U_i} : U_i \to k$ is given by $\left(P \mapsto \frac{g_i(P)}{h_i(P)}\right)$ and g_i is identically zero on $X \setminus U_i$.

Because the sets $D(h_i)$ for $1 \leq i \leq t$ cover X and h_i does not vanish on $D(h_i)$, we have that $h_1, \ldots, h_t$ cannot vanish simultaneously, that is $V(h_1, \ldots h_t) = \emptyset$ and the relative Nullstellensatz yields $(h_1, \ldots h_t) = k[X]$. Thus for some $\ell_1, \ldots, \ell_t \in k[X]$ we have

$$1 = \sum_{i=1}^{t} \ell_i h_i$$

Set $f = \sum_{i=1}^t \ell_i g_i \in k[X]$. We'll check that $\varphi = f$ as functions. In the mean time observe that on $D(h_i) \cap D(h_j)$ we have $\frac{g_i}{h_i} = \frac{g_j}{h_j}$ as both of these expressions are equal to φ. So $h_i g_j = h_j g_i$ on $D(f_i) \cap D(f_j)$. This equality also holds on $D(h_j) \setminus D(h_i)$ where $h_i = 0$ (by definition of $D(h_i)$) and $g_i = 0$ by Lemma 1.86.. Thus $h_i g_j = h_j g_i$ holds on $D(h_j)$.

Therefore on each $D(h_j)$ we have identities

$$\varphi = \frac{g_i}{h_i} = 1 \cdot \frac{g_j}{h_j} = \sum_{i=1}^t \ell_i h_i \cdot \frac{g_j}{h_j} = \sum_{i=1}^t \ell_i h_j \cdot \frac{g_i}{h_j} = \sum_{i=1}^t \ell_i g_i = f.$$

This completes the proof. □

Example 1.88. Suppose $U = D_X(h)$ is a distinguished open subset of an affine variety X for some $h \in k[X]$. Then every element of the ring

$$k[X]\,[1/h] = \left\{ \frac{g}{h^m} \mid g \in k[X] \right\}$$

determines a regular function on U.

This gives a ring map

$$\iota : k[X]\,[1/h] \to \mathcal{O}_X(D_X(h)), \quad \frac{g}{h^m} \mapsto \frac{g}{h^m}.$$

This map is also an isomorphism, so that the only regular functions on a distinguished open subset are the "obvious ones". The proof is similar to the previous theorem; we omit it.

Theorem 1.89. *Let X be an affine variety in $\mathbb{A}_k^n$ and let $0 \neq h \in k[X]$. The canonical map*

$$k[X]\,[1/h] \to \mathcal{O}_X(D_X(h))$$

is an isomorphism of k-algebras.

Wednesday, February 5 2025

We can deduce a localization formula for the ring of regular functions at $\mathbf{a}$. In particular this is a local ring (has a unique maximal ideal).

Corollary 1.90. *The regular functions at a point $\mathbf{a} \in X$ are the elements of*

$$\mathcal{O}_{X,\mathbf{a}} := k[X]_{I(\mathbf{a})} = \left\{ \frac{g}{h} \mid g, h \in k[X], h \notin I(\mathbf{a}) \right\}.$$

Proof sketch. We have that

$$k[X]_{I(\mathbf{a})} = \bigcup_{f \in k[X], f(\mathbf{a}) \neq 0} k[X][1/f] = \bigcup_{\mathbf{a} \in D(f)} \mathcal{O}_X(D(f)) = \bigcup_{\mathbf{a} \in U} \mathcal{O}_X(U).$$

The last equality is true since the open sets $D(f)$ are a basis for the topology of X. Finally the last set comprises the regular functions at $\mathbf{a}$. □

So far we have identified several rings of regular functions with localizations of the coordinate ring $k[X]$. But not every $\mathcal{O}_X(U)$ is such as localization. On non-distinguished open subsets there are interesting examples of regular functions:

Example 1.91. Let $X = V(xw - yz) \subseteq \mathbb{A}^4_k$ and U be the open subset of X given by

$$U = X\backslash V(y,w) = X\backslash(V(y)\cup V(w)) = \{(x,y,z,w) \in \mathbb{A}^4_k \mid xz = yw \text{ and } (y \neq 0 \text{ or } w \neq 0)\}.$$

Define $\alpha : U \to \mathbb{A}^1_k$ by the following rule: Set $U' = U \setminus V(y) = \{(x,y,z,w) \in U \mid y \neq 0\}$ and $U'' = U \setminus V(w) = \{(x,y,z,w) \in U \mid w \neq 0\}$, and note that $U = U' \cup U''$. Define

$$\alpha(x,y,z,w) = \begin{cases} \frac{x}{y} & \text{if } (x,y,z,w) \in U' \text{ and} \\ \frac{z}{w} & \text{if } (x,y,z,w) \in U''; \end{cases}$$

that is,

$$\alpha(x,y,z,w) = \begin{cases} \frac{x}{y} & \text{if } y \neq 0, \text{ and} \\ \frac{z}{w} & \text{if } w \neq 0. \end{cases}$$

Then α is a well-defined function, since whenever $y \neq 0$ and $w \neq 0$ (i.e., for points in $U' \cap U''$) we have $\frac{x}{y} = \frac{z}{w}$ thanks to the defining equation for X. Moreover, α is regular since $U = U' \cup U''$ and on each of U' and U'' it is given by a quotient of polynomials.

But α *cannot* be represented by any polynomial or even any rational function; that is, there do not exist $f, g \in k[X]$ such that $\alpha(P) = \frac{f(P)}{g(P)}$ for all $P \in U$. This means that α is not an element in any localization of $k[X]$.

The previous example brings up a relevant point: we can make interesting regular functions by glueing regular functions defined on distinguished open sets as long as they agree on the intersection.

Regular functions have the following properties which make them into a *sheaf* $\mathcal{O}_X$:

1. Every $\mathcal{O}_X(U)$ is a k-algebra.

2. If $U_1 \subseteq U_2$ are open sets the restriction map defines a k-algebra homomorphism $\text{res}_{U_2,U_1} : \mathcal{O}_X(U_2) \to \mathcal{O}_X(U_1)$. In particular if $U_1 \subseteq U_2 \subseteq U_3$ are open sets the restrictions compose nicely $\text{res}_{U_2,U_1} \circ \text{res}_{U_3,U_2} = \text{res}_{U_3,U_1}$.

3. If $f_1 \in \mathcal{O}_X(U_1), f_2 \in \mathcal{O}_X(U_2)$ satisfy $f_1|_{U_1\cap U_2} = f_2|_{U_1\cap U_2}$, then there is a unique $f \in \mathcal{O}_X(U_1 \cup U_2)$ so that $f|_{U_1} = f_1$ and $f|_{U_2} = f_2$.

An element $f \in \mathcal{O}_X(U)$ is then called a *section* of the sheaf $\mathcal{O}_X$. The ring $\mathcal{O}_{X,\mathbf{a}}$ is called the *stalk* of $\mathcal{O}_X$ at $\mathbf{a}$.

1.6 (Bi)-Rational maps

Next we will consider maps between irreducible varieties given componentwise by rational functions.

Definition 1.92. Let $X \subseteq \mathbb{A}_k^n$ and $Y \subseteq \mathbb{A}_k^m$ be irreducible affine varieties. A *rational map* $X \dashrightarrow \mathbb{A}_k^m$ is a function (the dashed arrow means it's only defined on an open subset of X)

$$\varphi : X \dashrightarrow \mathbb{A}_k^m, \varphi(\mathbf{a}) = (\varphi_1(\mathbf{a}), \dots, \varphi_m(\mathbf{a})) \text{ for some } \varphi_i \in k(X) = \mathrm{Frac}(k[X]).$$

The function φ is *regular at* $\mathbf{a}$ if each φ_i is regular at $\mathbf{a}$, that is, $\varphi_i(\mathbf{a})$ is defined.

A rational map $\varphi : X \dashrightarrow Y$ is a rational map $\varphi : X \dashrightarrow \mathbb{A}_k^m$ whose image lands in Y.

We would like to consider pullbacks of rational maps.

Definition 1.93. If $\varphi : X \dashrightarrow Y$ is a rational map, its *pullback* is the function

$$\varphi^* : k[Y] \to k(X), \varphi^*(u) = u \circ \varphi = u(\varphi_1, \dots, \varphi_m). \tag{1.1}$$

Note that the domain of the pullback above is regular functions. We would like to extend the pullback to rational functions $k(Y)$ as inputs. However, there are a few issues: first, if u is a rational function on X, the image of φ may land outside the domain of definition of u, so $u \circ \varphi$ may not make sense. Second, we would like to use the UMP of the fraction field to induce a map $\widetilde{\varphi^*} : k(Y) \to k(X), \widetilde{\varphi^*}(g/h) = \varphi^*(g)/\varphi^*(h)$ based on (1.1). However, for this to be defined we need $\varphi^*(h) \neq 0$ whenever $h \neq 0$. So we need φ^* to be injective.

Definition 1.94. A map $\varphi : X \dashrightarrow Y$ is *dominant* if $\mathrm{Im}(\varphi)$ is dense in Y, that is $\overline{\mathrm{Im}(\varphi)} = Y$.

Lemma 1.95. *If $\varphi : X \dashrightarrow Y$ is a rational map, the pullback $\varphi^* : k[Y] \to k(X)$ is injective if and only if φ is dominant.*

Proof. Suppose $\varphi^* : k[Y] \to k(X)$ is injective and suppose $\overline{\mathrm{Im}(\varphi)} \subseteq V(u)$ for some $u \in k[Y]$. Then $\varphi^*(u) = 0$ so $u = 0$, so $\overline{\mathrm{Im}(\varphi)} = V(0) = Y$.

Suppose φ is dominant and $u \in \mathrm{Ker}(\varphi^*)$. Then $\varphi^*(u) = 0$ so $u = 0$ on $\mathrm{Im}(\varphi)$, equivalently $\mathrm{Im}(\varphi) \subseteq V(u)$. This implies $\overline{\mathrm{Im}(\varphi)} = Y = V(0) \subseteq V(u)$ so $u = 0$. ☐

This leads to the following result whose proof is similar to the bijection between regular maps $f : X \to Y$and k-algebra homomorphisms $f^* : k[Y] \to k[X]$ in Theorem 1.52.

Proposition 1.96. *There is a bijection*

$$\begin{array}{ccc} \textit{dominant rational maps} & \leftrightarrow & \textit{inclusions of field extensions of } k \textit{ fixing } k \\ \varphi : X \dashrightarrow Y & \leftrightarrow & \varphi^* : k(Y) \to k(X) \end{array}$$

Definition 1.97. A pair of dominant rational maps $\varphi : X \dashrightarrow Y$ and $\psi : Y \dashrightarrow X$ so that $\varphi \circ \psi = \mathrm{id}_Y$ and $\psi \circ \varphi = \mathrm{id}_X$ wherever these compositions are defined are each called *birational*. If such maps exist, X and Y are called *birational*.

Corollary 1.98. *A dominant rational map $\varphi : X \dashrightarrow Y$ is birational if and only if its pullback is an isomorphism $\varphi^* : k(Y) \xrightarrow{\cong} k(X)$.*

Example 1.99 (Stereographic projection)**.** Consider the sphere $X = V(x^2+y^2+z^2-1)$ and $Y = V(z) \cong \mathbb{A}^2_k$. The map that projects from the north pole of the sphere $(0,0,1)$ onto the $z = 0$ plane is given by

$$\pi : \mathbb{A}^3_k \dashrightarrow \mathbb{A}^2_k, \quad \pi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

and has inverse

$$\pi^{-1} : \mathbb{A}^2_k \dashrightarrow \mathbb{A}^3_k, \quad \pi^{-1}(x,y) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, 1 - \frac{2}{1+x^2+y^2}\right).$$

These maps are birational.

Chapter 2

Projective Varieties

2.1 Projective space

Algebraic geometry works better over the natural compactification of $\mathbb{A}_k^n$, the projective space $\mathbb{P}_k^n$. For one example: in $\mathbb{A}_k^2$ two different lines can intersect at either one or zero points, whereas in $\mathbb{P}_k^2$ two different lines will always intersect in exactly one point. More generally, in $\mathbb{P}_k^2$ we'll be able to show facts such as

> [Bezout's theorem] In $\mathbb{P}_k^2$ over an algebraically closed field k, two curves of degrees d and e, that share no irreducible components meet at precisely $d \cdot e$ points, counted with multiplicity.

Definition 2.1 (Projective space)**.** For any field k, define the *projective n-space*

$$\mathbb{P}_k^n = \frac{\mathbb{A}^{n+1} \setminus \{(0, \dots, 0)\}}{\sim}$$

where $\sim$ is an equivalence relation defined by $(a_0, \dots, a_n) \sim (b_0, \dots, b_n)$ if and only if $(a_0, \dots, a_n) = l(b_0, \dots, b_n)$ for some $l \neq 0, l \in k$.

The equivalence class of $(a_0, \dots, a_n) \in \mathbb{A}^{n+1} \setminus \{(0, \dots, 0)\}$ is written as $[a_0 : \cdots : a_n]$ or $[\mathbf{a}]$ and referred to as a point in $\mathbb{P}_k^n$.

Remark 2.2. Two points $[a_0 : \cdots : a_n]$ and $[b_0 : \cdots : b_n]$ are equal if and only if the matrix

$$\begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \end{bmatrix}$$

has rank one if and only if all the 2×2 minors of the above matrix are equal to zero.

We could equivalently define $\mathbb{P}_k^n$ to be the set of lines through the origin in $\mathbb{A}_k^{n+1}$: given such a line, any non-origin point on it determines the same point in $\mathbb{P}_k^n$ as we have defined it.

We could also think of $\mathbb{P}_k^n$ as being obtained by taking $\mathbb{A}_k^n$ (not $\mathbb{A}^{n+1}$) and adding points at infinity.

Example 2.3. $\mathbb{P}^1_k$ consists of points $[a_0 : a_1]$ with either $a_0 \neq 0$ or $a_1 \neq 0$. Note that if $a_0 \neq 0$ then we have $[a_0 : a_1] = [1 : a_1/a_0]$. Consider the partially-defined function $\phi : \mathbb{P}^1_k \dashrightarrow \mathbb{A}^1_k$ sending $[a_0 : a_1] \mapsto a_1/a_0$. It is well-defined (i.e., independent of representative) and one-to-one everywhere it is defined, but it is undefined at $[0 : a_1]$ for $a_1 \neq 0$. It is clearly onto. Note that $[0 : a_1] = [0 : 1]$ for any $a_1 \neq 0$ and so ϕ is not defined at just one point: $[0 : 1]$. Thus, ϕ induces a bijection $\mathbb{P}^1_k \setminus \{[0 : 1]\} \xrightarrow{\cong} \mathbb{A}^1_k$ so we think of $\mathbb{P}^1_k$ as the affine line together with one point at infinity, namely $[0 : 1]$.

An inverse of the bijection $\mathbb{P}^1_k \setminus \{[0 : 1]\} \xrightarrow{\cong} \mathbb{A}^1_k$ is given by sending t to $[1 : t]$. For $t \neq 0$ we can rewrite this as $[1/t : 1]$. Now suppose $k = \mathbb{R}$ or $\mathbb{C}$ and let t go to infinity. The point $[1/t : 1]$ converges to $[0 : 1]$, as we would hope.

For $k = \mathbb{R}$, one can think of $\mathbb{P}^1_{\mathbb{R}}$ as a circle. For $k = \mathbb{C}$, think of $\mathbb{P}^1_{\mathbb{C}}$ as a sphere, the Riemann sphere. In each case it is the one-point compactification of $\mathbb{A}^1$ with its classical (Euclidean) topology.

Example 2.4. A point in $\mathbb{P}^2_k$ is $[a_0 : a_1 : a_2]$ with at least one $a_i \neq 0$. We may define an injection $i : \mathbb{A}^2_k \hookrightarrow \mathbb{P}^2_k$ by $i(a_1, a_2) = [1 : a_1 : a_2]$. The complement of its image is $\{[0 : a_1 : a_2] \mid a_1 \neq 0 \text{ or } a_2 \neq 0\}$. Note that this complement can be identified with $\mathbb{P}^1_k$.

An arbitrary line in $\mathbb{A}^2_k$ is given parametrically by $t \mapsto (ta_1 + c, ta_2 + c)$ for some choice of a_1, a_2 not both zero and any c. Applying i we obtain the parametric line $t \mapsto [1 : ta_1 + c : ta_2 + c]$ which for $t \neq 0$ can also be written as $t \mapsto [1/t : a_1 + c/t : a_2 + c/t]$. "Letting t go to ∞" gives the point $[0 : a_1 : a_2]$.

We thus think of $\mathbb{P}^2$ as being $\mathbb{A}^2_k$ with one point at infinity adjoined for each possible slope of a line. That is, any two parallel lines in $\mathbb{A}^2$ meet at a uniquely determined point at infinity, and any two non-parallel lines do not meet at infinity.

When $k = \mathbb{R}$, $\mathbb{P}^2_{\mathbb{R}}$ is the Eulicidean plane with a "circle of points" at infinity. It can be visualized by starting with the unit disc and identifying antipodal points. The resulting surface is *non-orientable.*

In general we have

$$\mathbb{P}^n_k = \mathbb{A}^n_k \cup \mathbb{P}^{n-1}_k = \cdots = \mathbb{A}^n_k \cup \mathbb{A}^{n-1}_k \cup \cdots \cup \mathbb{A}^1_k \cup \{\infty\}.$$

In coordinates this looks like

$$[a_0 : \cdots : a_n] \leftrightarrow \begin{cases} \left(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}\right) \in \mathbb{A}^n_k, & \text{if } a_0 \neq 0 \\ [a_1 : \cdots : a_n] \in \mathbb{P}^{n-1}, & \text{if } a_0 = 0. \end{cases}$$

The choice of a_0 above is arbitrary. Set

$$U_i = \{[a_0 : \cdots : a_n] \mid a_i \neq 0\}.$$

Then $\mathbb{P}^n_k = \bigcup_{i=0}^n U_i$ with each of the sets U_i in bijection with $\mathbb{A}^n_k$ via

$$\Psi_i : U_i \to \mathbb{A}^n_k, [a_0 : \cdots : a_n] \mapsto \left(\frac{x_0}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}\right).$$

When $k = \mathbb{R}$ or $k = \mathbb{C}$, $\mathbb{P}^n_k$ inherits the Euclidean quotient topology. In this topology the U_i form an atlas of open sets turning P^n_k into a manifold. We'll see that these sets are also open in the Zariski topology as well (later).

Friday, February 7

2.2 Projective varieties

2.2.1 Graded rings

A (commutative) ring R is $\mathbb{N}$-*graded* (respectively, $\mathbb{Z}$-graded) if its underlying abelian group under $+$ is endowed with an internal direct sum decomposition $R = \bigoplus_{i \in \mathbb{N}} R_i$ (resp., $R = \bigoplus_{i \in \mathbb{Z}} R_i$) such that if $x \in R_i$ and $y \in R_j$ then $xy \in R_{i+j}$.

Elements of R_i are known as *homogenous elements* of *degree* i. Note that every element r or R admits a unique decomposition as $r = \sum_i r_i$ (finite sum) with r_i homogeneous of degree i.

An *homogeneous ideal* I of a graded ring R is an ideal in the usual sense (closed under addition and scaling) such that the following equivalent properties hold:

1. I is generated by homogenous elements (not necessarily all of the same degree),
2. for each $a \in I$, if $a = \sum_j a_j$ with $a_j \in R_j$, we have $a_j \in I$ for all j, or
3. $I = \bigoplus I_j$ is the internal direct sum of $I_j := I \cap R_j$, $j \in \mathbb{Z}$.

If I is a homogenous ideal in a graded ring R, then the quotient ring R/I is canonically graded, by setting $(R/I)_j := \{x + I \mid x \in R_j\} \cong R_j/I_j$, and the canonical map $R \twoheadrightarrow R/I$ preserves the grading.

Example 2.5. The standard $\mathbb{N}$-grading on $R = k[X_0, \ldots, X_n]$ is given by letting R_j be all k-linear combinations of monomials of degree j, where a monomial refers to an element of the form $X_0^{e_0} \cdots X_n^{e_n}$ with $e_i \geq 0$ and its degree is $\sum_i e_i$. So, a polynomial $f \in R$ is homogenous of deegree d if it has the form $\sum_{e_0,\ldots,e_n; e_0+\cdots+e_n=d} l_{e_0,\ldots,e_n} X_0^{e_0} \cdots X_n^{e_n}$.

Remark 2.6. If I, J are homogeneous ideals then so are $I + J$, IJ, $I \cap J$, $I : J$ and the associated primes of I.

An issue we have in projective space is that polynomials do not give well-defined functions on $\mathbb{P}^n_k$. However the vanishing set of a homogeneous polynomials is well-defined.

Lemma 2.7. *Suppose $f \in k[X_0, \ldots, X_n]$ is homogeneous. If f vanishes for one choice of representatives for a point $[a_0 : \cdots : a_n] \in \mathbb{P}^n_k$, then it vanishes for all choices of representatives.*

Proof. Suppose $f(X_0, \ldots, X_n) = \sum_{e_0,\ldots,e_n; e_0+\cdots+e_n=d} c_{e_0,\ldots,e_n} X_0^{e_0} \cdots X_n^{e_n}$. We have that $(la_0)^{e_0} \cdots (la_n)^{e_n} = l^{\sum_i e_i} a_0^{e_0} \cdots a_n^{e_n}$, so we conclude

$$f(la_0, \ldots, la_n) = l^d f(a_0, \ldots, a_n).$$

Thus $f(la_0, \ldots, la_n) = 0$ iff $f(a_0, \ldots, a_n) = 0$. $\square$

The geometric interpretation of the Lemma is that if f is homogenous, then the affine variety $V(f) \subseteq \mathbb{A}_k^{n+1}$ is a union of lines passing through the origin. In other words, it is a *cone*. More generally, if I is a homogeneous ideal then the affine variety $V(I)$ is a cone.

2.2.2 Projective Varieties

Lemma 2.7 shows that the following definition makes sense.

Definition 2.8. Given a homogenous polynomial $f \in k[X_0, \dots, X_n]$, let

$$V^{\mathbb{P}}(f) = \{[a_0 : \dots : a_n] \in \mathbb{P}_k^n \mid f(a_0, \dots, a_n) = 0\} \subseteq \mathbb{P}_k^n.$$

More generally, if $I \subseteq k[X_0, \dots, X_n]$ is a homogeneous ideal, define

$$V^{\mathbb{P}}(I) = \{[a_0 : \dots : a_n] \in \mathbb{P}_k^n \mid f(a_0, \dots, a_n) = 0, \forall f \in I \text{ homogeneous}\} \subseteq \mathbb{P}_k^n.$$

A *projective variety* is a subset of $\mathbb{P}_k^n$ for some n of the form $V^{\mathbb{P}}(I)$ for some homogenous ideal $I \subseteq k[X_0, \dots, X_n]$.

Geometrically, $V^{\mathbb{P}}(I)$ is the quotient of the punctured affine cone $V^{\mathbb{A}}(I)$ under $\sim$

$$V^{\mathbb{P}}(I) = V^{\mathbb{A}}(I) \setminus \{(0, \dots, 0)\} / \sim . \tag{2.1}$$

Let us look at some examples now.

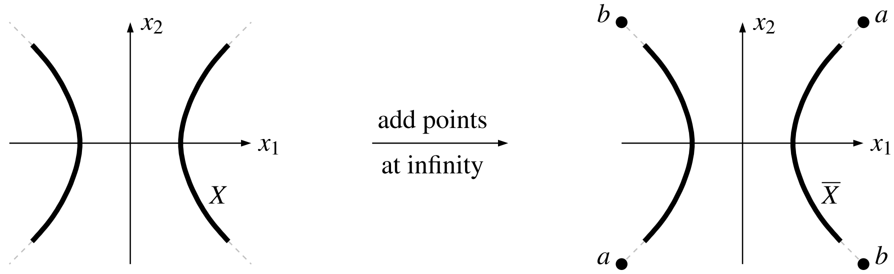
Example 2.9.
- $\mathbb{P}_k^n = V^{\mathbb{P}}((0)$ is a projective variety;
- $\emptyset = V((1))$ is an projective variety;
- Any singleton $\{[a_0 : \dots : a_n]\}$ is a projective variety, since it is the zero locus of the ideal of 2×2 minors of the matrix below, which are homogenous of degree one (linear forms)
$$\begin{bmatrix} X_0 & X_1 & \cdots & X_n \\ a_0 & a_1 & \cdots & a_n \end{bmatrix};$$
- Linear subspaces of k^n are affine varieties;
- Given a homogeneous polynomial f, the projective variety $V^{\mathbb{P}}(f)$ is called a *projective hypersurface*;
- Given a linear polynomial $\ell = c_0X_0 + c_1X_1 + \cdots c_nX_n$ with $c_i \in k$ (no constant term), the projective variety $V^{\mathbb{P}}(\ell)$ is called a *projective hyperplane*;

Our focus will be on thinking of projective varieties as being affine varieties with points added at infinity. Henceforth we shall write $V^{\mathbb{A}}$ for what was previously written as V (affine variety), to avoid confusion.

Example 2.10. Let $f = X^2 - Y^2 - Z^2$. The affine zero locus of this is the classical (double) cone in $\mathbb{A}_k^3$. Let $X = V^{\mathbb{P}}(f) \subseteq \mathbb{P}^2$ be the associated projective variety. By (2.1) at least over $k = \mathbb{R}$ this ought to be a circle.

Let us intersect X with the copy of $\mathbb{A}^2_k$ contained in $\mathbb{P}^2$ given by $U_2 = \{[a : b : c] \mid c \neq 0\}$. We have $X \cap U_2 = V^{\mathbb{A}}(x^2 - y^2 - 1) \subseteq \mathbb{A}^2$, which over $k = \mathbb{R}$ is a hyperbola. Since $X \cap (\mathbb{P}^2_k \setminus U_2) = \{[1 : 1 : 0], [1 : -1 : 0]\}$, we may think of this hyperbola as having two points at infinity. Note that upon adding these points, we do indeed get a circle.

In the picture below $x_1 = X, x_2 = Y$ and the points at infinity are marked as a and b. Note that the two points marked as a are actually identical and the two points marked as b are also identical since a is the unique "point at ∞" on the line $x_1 = x_2$ and b is the unique "point at ∞" on the line $x_1 = -x_2$.



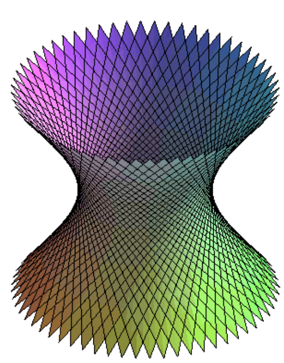
We may also consider $X \cap U_0 = V^{\mathbb{A}}(1 - y^2 - z^2)$, which is already a circle in the real affine plane. Where are the points at infinity? They are imaginary: $[0 : 1 : i]$ and $[0 : 1 : -i]$.

Example 2.11 (The smooth quadric surface)**.** In mathematics, a quadric or quadric hypersurface is the vanishing locus of a polynomial equation of degree 2. It turns out (because symmetric matrices can be diagonalized over an algebraically closed field) that after making linear changes of coordinates there is a unique smooth quadric given by the equation

$$F = X_0^2 + X_1^2 + \cdots + X_n^2.$$

We'll study the case $n = 3$ and take instead the equation to be $F = XY - ZW$. Note that this can be put into the form $X^2 + Y^2 - Z^2 - W^2$ by substituting $X \mapsto X + iY, Y \mapsto X - iY, Z \mapsto Z + iW, W \mapsto Z - iW$.

A picture of $V^{\mathbb{P}}(F) \cap U_3$ is included below. For $k = \mathbb{R}$ and after the substitution described above it is the hyperboloid $V^{\mathbb{A}}(x^2 + y^2 - z^2 - 1)$, which is known to be a ruled surface. Specifically, there are two families of lines on this surface as pictured. We will explain this later by providing an isomorphism $V^{\mathbb{P}}(XY - ZW) \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$.



2.2.3 Homogenization and dehomogenization

Definition 2.12. Given a not-necessarily homogenous polynomial $f(x_1,\dots,x_n) \in k[x_1,\dots,x_n]$ of degree d (i.e., the highest degree of any of its non-zero terms is d) define its *homogenization* to be the homogenous polynomial $f^h \in k[X_0,\dots,X_n]$ given by

$$f^h = X_0^d f(X_1/X_0,\dots,X_n/X_0).$$

Given a homogeneous polynomial $F \in k[X_0,\dots,X_n]$, the (usually) non-homogenous polynomial $f(x_1,\dots,x_n) = F(1,x_1,\dots,x_n)$ is called the *dehomogenization* of F.

Example 2.13. The homogenization of $y^2 - x^2(x+1) = y^2 - x^3 - x^2$ is $Y^2Z - X^2(X+Z) = Y^2Z - X^3 - X^2Z$. The homogenization of $y^2 - x^2$ is $Y^2 - X^2$, technically viewed as belonging to $k[X,Y,Z]$.

In general, the homogenization amounts to tacking on the smallest possible powers of a new variable to each term in order to make it homogenous. Beware: While the dehomogenization of the homogenization of f is f, the opposite can fail. For instance, the homogenization of the dehomogenization of $F = X_0^4 + X_1X_0^3 + X_2^2X_0^2$ is $X_0^2 + X_1X_0^1 + X_2^2 \neq F$. One does get equality so long as X_0 does not divide F.

Definition 2.14. Given an ideal $I \subseteq k[x_1,\dots,x_n]$ its *homogenization* is the ideal

$$I^h = (f^h \mid f \in I).$$

Warning: it is not true that if $I = (f_1,\dots,f_r)$ then $I^h = (f_1^h,\dots,f_r^h)$.

Example 2.15 (The projective twisted cubic)**.** Let $I = (x_1^2 - x_2, x_1^3 - x_3)$ be the ideal of the affine cubic curve. Then $x_1x_2 - x_3$ and $x_2^2 - x_1x_3$ are elements of I and

$$I^h = (X_1^2 - X_0X_2, X_1^3 - X_0^2X_3, X_1X_2 - X_0X_3, X_2^2 - X_1X_3).$$[1]

In fact the second generator above is redundant, so

$$I^h = (X_1^2 - X_0X_2, X_1X_2 - X_0X_3, X_2^2 - X_1X_3)$$

is generated by the maximal minors of

$$\begin{bmatrix} X_0 & X_1 & X_2 \\ X_1 & X_2 & X_3 \end{bmatrix}.$$

Proposition 2.16. *If I^h is the homogenization of $I \subseteq k[x_1,\dots,x_n]$, then $V^{\mathbb{P}}(I^h) \cap U_0$ corresponds to $V^{\mathbb{A}}(I)$ under the bijection*

$$\Psi_0 : U_0 \xrightarrow{\cong} \mathbb{A}_k^n, \qquad \Psi_0([a_0 : \dots : a_n]) = (a_1/a_0,\dots,a_n/a_0)$$

whose inverse is

$$\Psi_0^{-1} : \mathbb{A}_k^n \xrightarrow{\cong} U_0 \qquad \Psi_0^{-1}(b_1,\dots,b_n) = [1 : b_1 : \dots : b_n].$$

[1]This can be computed using Gröbner bases. See *Ideals, varieties, and algorithms* by Cox, Little and O'shea, Theorem 4, p. 388.

Proof. We have

$$\begin{aligned}(b_1,\dots,b_n)\in V^{\mathbb{A}}(I) &\iff f(b_1,\dots,b_n)=0,\forall f\in I\\ &\iff f^h(1,b_1,\dots,b_n)=0\\ &\iff [1:b_1:\cdots:b_n]\in V^{\mathbb{P}}(I^h)\cap U_0.\end{aligned}$$

The result follows since $\Psi_0^{-1}(b_1,\dots,b_n)=[1:b_1:\cdots:b_n]$. □

Definition 2.17. The projective variety $V^{\mathbb{P}}(I^h)\subseteq\mathbb{P}^n_k$ is called the *projective closure* of $V^{\mathbb{A}}(I)\subseteq\mathbb{A}^n_k$.

Monday, February 10

2.3 Projective Nullstellensatz

We have previously associagted a subset of $\mathbb{P}^n_k$ to each homogeneous ideal of $k[X_0,\dots,X_n]$. Now we do the opposite:

Definition 2.18. Given any subset W of $\mathbb{P}^n_k$, set

$$I^{\mathbb{P}}(W)=(f\in k[X_0,\dots,X_n]\text{ homogenous }\mid f(a_0,\dots,a_n)=0,\ \forall[a_0:\cdots:a_n]\in W)\,,$$

that is, the *ideal generated by* the homogeneous polynomials that vanish at all points of W. Note that $I^{\mathbb{P}}(W)$ is a homogeneous ideal of $k[X_0,\dots,X_n]$ since, by construction, it is generated by homogenous elements.

Exercise 2.19. Show that

$$I^{\mathbb{P}}(W)=\{f_1+\cdots+f_m\mid f_i\text{ homogeneous, }f_i(a_0,\dots,a_n)=0,\ \forall i\text{ and all }[a_0:\cdots:a_n]\in W\}.$$

Exercise 2.20. Show $I^{\mathbb{P}}(W)$ is a radical ideal.

In projective space there is a very troublesome ideal. This corresponds to the origin which we removed from $\mathbb{A}^{n+1}_k$ to form $\mathbb{P}^n_k$.

Definition 2.21. The homogenous ideal $\mathfrak{m}=(X_0,\dots,X_n)$ is called the *irrelevant ideal* of $k[X_0,\dots,X_n]$.

Note that the irrelevant ideal is a homogeneous radical ideal (in fact it's prime) and satisfies $V(X_0,\dots,X_n)=\emptyset=V(1)$. So we cannot hope for a bijection between all homogeneous radical ideals and projective varieties. To get a true bijection we have to remove the irrelevant ideal as follows.

Definition 2.22. Let I be an ideal of $k[X_0,\dots,X_n]$. Its *saturation* I^{sat} with respect to the irelevant idela $\mathfrak{m}=(X_0,\dots,X_n)$ is the ideal

$$I^{\text{sat}}=\bigcup_{i\geq 1}I:\mathfrak{m}^i=\{f\in k[X_0,\dots,X_n]\mid f\cdot\mathfrak{m}^i\subseteq I\text{ for some }i\geq 1\}.$$

Exercise 2.23. Show that if $I = \bigcap_{i=1}^{m} Q_i$ is a primary decomposition with Q_m $\mathfrak{m}$-primary, then $I^{\mathrm{sat}} = \bigcap_{i=1}^{m-1} Q_i$. Deduce that $V^{\mathbb{P}}(I^{\mathrm{sat}}) = V^{\mathbb{P}}(I)$.

We can now characterize all the ideals whose projective vanishing set is empty.

Proposition 2.24. *For a homogeneous ideal $I \subseteq S = k[X_0, \ldots, X_n]$ TFAE*

1. $V^{\mathbb{P}}(I) = \emptyset$
2. $\sqrt{I} \in \{(1), \mathfrak{m}\}$
3. $S_d \subseteq I$ *for some* $d \geq 1$, *(S_d is the set of homogeneous polynomials of degree d)*
4. $I^{\mathrm{sat}} = S$.

Exercise 2.25. Prove the proposition.

Theorem 2.26 (Projective Nullstellensatz)**.** *For an algebraically closed field k, the functions $V^{\mathbb{P}}$ and $I^{\mathbb{P}}$ determine an order-reversing bijection between the partially ordered sets (with respect to containment)*

$$\text{Projective Varieties in } \mathbb{P}^n_k \xleftrightarrow[I^{\mathbb{P}}]{V^{\mathbb{P}}} \text{homogenous, radical ideals in } k[X_0, \ldots, X_n]$$
$$\text{that are not equal to the irrelevant ideal.}$$

That is,

$$V^{\mathbb{P}}(I^{\mathbb{P}}(X)) = X \qquad \text{and} \qquad I^{\mathbb{P}}(V^{\mathbb{P}}(J)) = J$$

for X any projective variety and J any homogenous, radical ideal other than the irrelevant one.

Remark 2.27. Note $I^{\mathbb{P}}(V^{\mathbb{P}}((X_0, \ldots, X_n))) = k[X_0, \ldots, X_n]$.

Proof of the Nullstellensatz. Let me just prove the hardest part (and the only part, in fact, that requires k to be algebraically closed): $I^{\mathbb{P}}(V^{\mathbb{P}}(J)) \subseteq J$ for a radical ideal J that is not equal to the irrelevant one. We'll do so by reducing to the affine Nullstellensatz (Theorem 1.43).

If $J = (1)$, the claimed containment is true because $\mathbb{V}^{\mathbb{P}}(1) = \emptyset$ and $I^{\mathbb{P}}(\emptyset) = (1)$. Denote $\mathfrak{m} = (X_0, \ldots, X_n)$. Now assume J is a proper, homogeneous, radical ideal and that $J \neq \mathfrak{m}$. Since J is homogeneous and all non-constant homogeneous polynomials belong to $\mathfrak{m}$, it follows that $J \subsetneq \mathfrak{m}$. In particular, since the containment is proper, by the affine Nullstellensatz, we have $V^{\mathbb{A}}(J) \supsetneq \{(0, \ldots, 0)\}$. That is, the affine cone $V^{\mathbb{A}}(J)$ contains at least one point other than origin (and hence it contains the entire line through the origin and such a point).

Pick a homogenous element $f \in I^{\mathbb{P}}(V^{\mathbb{P}}(J))$. Then $f([a_0 : \cdots : a_n]) = 0$ for all $[a_0 : \cdots : a_n] \in V^{\mathbb{P}}(J)$. It follows that $f(a_0, \ldots, a_n) = 0$ for all points $(a_n, \ldots, a_0) \in V^{\mathbb{A}}(J) \setminus \{(0, \ldots, 0)\}$. As $f \in J \subsetneq \mathfrak{m}$, f cannot be a non-zero constant; in particular, since f is homogeneous and non-constant, $f(0, \ldots, 0) = 0$. We have shown $f \in I(V^{\mathbb{A}}(J))$, and thus by the classical Nullstellensatz we have $f \in J$.

We have shown that every homogenous element of $I^{\mathbb{P}}(V^{\mathbb{P}}(J))$ belongs to J. Since $I^{\mathbb{P}}(V^{\mathbb{P}}(J))$ is a homogenous ideal, it is generated by its homogeneous elements, and thus $I^{\mathbb{P}}(V^{\mathbb{P}}(J)) \subseteq J$. □

2.4 Zariski topology

Definition 2.28. The *Zarsiki topology* on $\mathbb{P}^n_k$ is the topology whose closed subsets are the projective varieties contained in $\mathbb{P}^n_k$ and the open subsets are the complements of projective varieties. This is a topology since

1. $\mathbb{P}^n_k = V^{\mathbb{P}}(0)$,
2. $\emptyset = V^{\mathbb{P}}(k[X_0, \dots, X_n])$,
3. $\bigcap_\alpha V^{\mathbb{P}}(I_\alpha) = V^{\mathbb{P}}(\sum_\alpha I_\alpha)$ for any collection $\{I_\alpha\}$ of homogeneous ideas (and an arbitrary sum of homogeneous ideals is again homogeneous), and
4. $V^{\mathbb{P}}(I) \cup V^{\mathbb{P}}(J) = V^{\mathbb{P}}(I \cap J)$ (and an intersection of homogenous ideals is again homogeneus).

Each of these can be justified by by considering the analgous facts for affine cones.

More generally, for a projective variety $W \subseteq \mathbb{P}^n_k$, the *Zariski topology on* W is the inherited subspace topology. If $W = V^{\mathbb{P}}(I)$ for a homogeneous radical ideal I, then the closed subsets of W are given by subsets of the form $V^{\mathbb{P}}(J)$ where J us a homogenous radical ideal such that $J \subseteq I$.

Example 2.29. The Zariski topology on $\mathbb{P}^1_k$ is the "finite complement" topology: A set is open if and only if it is empty or is it the complement of a finite subset of $\mathbb{P}^1$. This holds since, as we saw above, every one element subset is closed and hence every finite subset is closed. If $F(X, Y)$ is any homogeneous polynomial in two variables, then F factors as $\prod_i (b_i X - a_i Y)$ for some $a_i, b_i \in k$, and thus the zero locus of F is the finite set $\{[a_1 : b_1], \dots, [a_d : b_d]\}$. This proves that every closed subset of $\mathbb{P}^1$, other than $\mathbb{P}^1_k$ itself, is finite.

Definition 2.30. Let $F \in k[X_0, \dots, X_n]$ be a homogenous polynomial. Let $D^{\mathbb{P}}(F)$ denote the open subset

$$D^{\mathbb{P}}(F) = \mathbb{P}^n_k \setminus V^{\mathbb{P}^n}(F) = \{[a_0 : \cdots a_n] \in \mathbb{P}^n_k \mid F(a_0, \dots, a_n) \neq 0\}.$$

We call any such an open subset *distinguished*. More generally for a projective variety $W \subseteq \mathbb{P}^n$, we write $D^{\mathbb{P}}_W(F)$ for the distinguished open subset $W \setminus V^{\mathbb{P}^n}(F)$.

As with distinguished open subsets in the affine setting,

Proposition 2.31. *The distinguished open subsets of a projective variety form a basis for the Zariski topology.*

Proof. This holds since $D^{\mathbb{P}}(F) \cap D^{\mathbb{P}}(G) = D^{\mathbb{P}}(F \cdot G)$ (note that the product of two homogenous polynomials is again homogeneous). Moreover, every closed subset has the form $Z = V^{\mathbb{P}}(F_1, \dots, F_m)$ for homogeneous polynomials $F_1, \dots, F_m$ and $V^{\mathbb{P}}(F_1, \dots, F_m) = \cap_j V^{\mathbb{P}}(F_j)$ so that $\mathbb{P}^n \setminus Z = \cup_j D^{\mathbb{P}}(F_j)$. That is, every open subset is a finite union of distinguished ones, just as in the affine case. □

Example 2.32. For any i, $D^{\mathbb{P}}(X_i)$ is the open subset $\{[a_0 : \cdots : a_n] \in \mathbb{P}^n_k \mid a_i \neq 0\}$. This is the subset I wrote as U_i before, so $U_i = D^{\mathbb{P}}(X_i)$.

Recall that for any i with $0 \leq i \leq n$ there is a bijection

$$\Psi_i : \mathbb{A}^n_k \to U_i, \text{ given by } \Psi(b_1, \dots, b_n) = [b_1 : \cdots : b_{i-1} : 1 : b_i : \cdots : b_n] \tag{2.2}$$

with inverse given by

$$\Psi_i^{-1}([a_0 : \cdots : a_n]) = (a_0/a_i, \dots, \widehat{a_i/a_i}, \dots, a_n/a_i). \tag{2.3}$$

Proposition 2.33. *The map Ψ_i in (2.2) is a homeomorphism (continuous with continuous inverse) for each i, where $\mathbb{A}^n_k$ is the usual affine Zariski topology and U_i is given the subspace topology of the Zariski topology on $\mathbb{P}^n_k$.*

Proof. For notational simplicity, assume $i = 0$. Since the distinguished open sets form a basis for either topology, it suffices to show distinguished open subsets correspond to each other under the bijection Ψ_0. Thanks to Proposition 2.16, given $f \in k[t_1, \dots, t_n]$ the open subset $D(f) \subseteq \mathbb{A}^n_k$ corresponds, under ψ_0, to the open subset $D^{\mathbb{P}}(F) \cap U_0$ of U_0, where $F = X_0^{\deg(f)}(f(X_1/X_0, \dots, X_n/X_0))$, the homogenization of f. Similarly, thanks to Proposition 2.16, given any homogeneous polynomial F, the open subset $D^{\mathbb{P}}(F) \cap U_0$ of U_0 corresponds, under Ψ_0, to $D^{\mathbb{A}}(f)$ where $f(t_1, \dots, t_n) := F(1, t_1, \dots, t_n)$ is the dehomogenization of F. In symbols, we have:

$$\begin{aligned} \Psi_0^{-1}(D^{\mathbb{P}}(F) \cap U_0) &= D^{\mathbb{A}}(F(1, t_1, \dots, t_n)) \\ (\Psi_0^{-1})^{-1}(D^{\mathbb{A}}(f)) &= D^{\mathbb{P}}(f^h). \end{aligned}$$

Because preimage commutes with union, and arbitrary open subsets are unions of distinguished ones, this suffices to finish the proof. □

So, $\mathbb{P}^n_k$ is the union of $n+1$ open subsets, $U_0, \dots, U_n$, each of which is homeomorphic to affine n-space.

Wednesday, February 12 – snow day

Friday, February 14

2.5 Regular functions and regular maps

Definition 2.34. Let $X \subseteq \mathbb{P}^n_k$ be a projective variety. A function $f : X \to \mathbb{A}^k_1$ is a *regular function* if $f|_{X \cap U_i}$ is a regular function on each set $X \cap U_i$ identified with a subset of A^n_k via the map $\Psi_i : U_i \to \mathbb{A}^n_k$ (2.2). This means that for each $0 \leq i \leq n$ and for each point $[\mathbf{a}] \in X \cap U_i$ there exists an open set U' so that $[\mathbf{a}] \in U' \subseteq X \cap U_i$ and polynomials

$$g, h \in k[x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n] \text{ such that } f([\mathbf{a}]) = \frac{g(\psi_i([\mathbf{a}]))}{h(\psi_i([\mathbf{a}]))} \text{ on } U'$$

(this includes the fact that the denominator must be non-zero).

Homogenizing g and h using the variable X_i to polynomials $\tilde{g}$ and $\tilde{h}$ of the same degree, that is for $d = \max\{\deg(g), \deg(h)\}$ denoting

$$\begin{aligned}\tilde{g}(X_0,\ldots,X_n) &= X_i^d g(X_0/X_i,\ldots,X_{i-1}/X_i,\ldots,X_n/X_i)\\ \tilde{g}(X_0,\ldots,X_n) &= X_i^d g(X_0/X_i,\ldots,X_{i-1}/X_i,\ldots,X_n/X_i).\end{aligned}$$

we can re-write

$$f([\mathbf{a}]) = \frac{\tilde{g}([\mathbf{a}])}{\tilde{h}([\mathbf{a}])} \text{ on } U'.$$

So, an equivalent definition is that a function $f : X \to \mathbb{A}_1^k$ is a *regular function* on X if for each point $[\mathbf{a}] \in X$ there exists an open set U' of X with $[\mathbf{a}] \in U'$ and $\tilde{g}, \tilde{h} \in k[X_0,\ldots,X_n]$ homogeneous polynomials of the same degree so that $f = \tilde{g}/\tilde{h}$ on U' (this includes the stipulation that $\tilde{h}$ does not vanish on U'.

Definition 2.35. Let $X \subseteq \mathbb{P}_k^n$ be a projective variety. A function $f : X \to \mathbb{P}_m^k$ is a *regular map* or *morphism* of projective varieties if its components, namely the functions $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i \cong \mathbb{A}_k^m$ are regular functions, that is

$$f([\mathbf{a}]) = \left[\frac{g_0([\mathbf{a}])}{h_0([\mathbf{a}])} : \cdots : \frac{g_{i-1}([\mathbf{a}])}{h_{i-1}([\mathbf{a}])} : 1 : \frac{g_{i+1}([\mathbf{a}])}{h_{i+1}([\mathbf{a}])} : \cdots : \frac{g_m([\mathbf{a}])}{h_m([\mathbf{a}])}\right]$$

on some Zariski open set U of X so that $\mathbf{a} \in U$ and $g_i, h_i \in k[X_0,\ldots,X_n]$ are homogeneous polynomials with $\deg(g_i) = \deg(h_i)$ and $h_i([\mathbf{a}]) \neq 0$ for all $0 \leq i \leq n$.

Clearing denominators (multiplying by $\prod h_i([\mathbf{a}])$) we can write

$$f([\mathbf{a}]) = [f_0([\mathbf{a}]) : \cdots : f_{i-1}([\mathbf{a}]) : f_i([\mathbf{a}]) : f_{i+1}([\mathbf{a}]) : \cdots : f_n([\mathbf{a}])] \text{ on } U \tag{2.4}$$

where $f_0,\ldots,f_n \in k[X_0,\ldots,X_n]$ are homogeneous polynomials of the same degree (equal to the degree of $\prod h_i([\mathbf{a}])$) so that $V^{\mathbb{P}}(f_0,\ldots,f_n) \cap U = \emptyset$.

So, an equivalent definition of a regular map $f : X \to \mathbb{P}_m^k$ is a function as in (2.4). A regular map $f : X \to Y$, where $Y \subseteq \mathbb{P}_k^m$ is a projective variety, is a regular map $f : X \to \mathbb{P}_m^k$ whose image is contained in Y.

Exercise 2.36. Prove that for $m, n > 0$ every regular map $f : \mathbb{P}_k^n \to \mathbb{P}_k^m$ is defined globally, i.e. it must be of the form

$$f = [f_0 : f_1 : \cdots : f_m] \text{ with } f_0,\ldots,f_m \in k[X_0,\ldots,X_n]$$

homogeneous polynomials of the same degree such that $V^{\mathbb{P}}(f_0,\ldots,f_m) = \emptyset$.

Definition 2.37. Projective varieties X and Y are *isomorphic* provided there are regular maps $f : X \to Y$ and $g : Y \to X$ that are inverse to each other. In this case f and g are called projective isomorphisms.

Example 2.38. Let $A \in \mathrm{GL}_{n+1}(k)$ and consider the regular map $\varphi_A : \mathbb{P}^n_k \to \mathbb{P}^n_k$

$$\varphi_A([a_0 : \ldots : a_n]) = A\mathbf{a}^T = \left[\sum_{j=0}^n a_{0j}X_j : \sum_{j=0}^n a_{1j}X_j : \ldots, \sum_{j=0}^n a_{nj}X_j\right].$$

Since A is invertible, φ_A is invertible too with regular inverse $\varphi_{A^{-1}}$. Thus φ_A is an isomorphism.

It turns out all automorphisms of $\mathbb{P}^n_k$ are of this form.

Exercise 2.39. Let $\mathrm{PGL}_{n+1}(K) = \mathrm{GL}_{n+1}(k)/\sim$ where $A \sim B$ if and only if $B = \lambda A$ for some $0 \neq \lambda \in k$. Show that $\mathrm{Aut}(\mathbb{P}^n_k) \cong \mathrm{PGL}_{n+1}(K)$ as groups.

We call the map φ_A in Example 2.38 a *linear change of coordinates* on $\mathbb{P}^n_k$.

Definition 2.40. The projective subvarieties X, Y of $\mathbb{P}^n_k$ are *projectively equivalent* if there is a linear change of coordinates φ_A such that $\varphi_k[X] = Y$.

Example 2.41. $V^{\mathbb{P}}(X_0)$ is projectively equivalent to $V^{\mathbb{P}}(X_i)$ by setting A to be the matrix that implements the transposition of the 0-th and i-th coordinates.

Inspired by the coordinate ring of an affine variety we define

Definition 2.42. The *homogeneous coordinate ring* of a projective variety $X \subseteq \mathbb{P}^n_k$ is

$$k^{\mathbb{P}}[X] = \frac{k[X_0, \ldots, X_n]}{I^{\mathbb{P}}(X)}.$$

Since $I^{\mathbb{P}}(X)$ is a homogeneous ideal, $k^{\mathbb{P}}[X]$ is a graded ring. Warning: not every element of $k^{\mathbb{P}}[X]$ is a well-defined function on X; only the homogeneous elements are well-defined as functions on X.

Warning: the coordinate ring is no longer helpful in determining whether two projective varieties are isomorphic. **In particular, an isomorphism of projective varieties need not induce an isomorphism between their coordinate rings.** However projective equivalence of varieties does lead to an isomorphism of coordinate rings.

Exercise 2.43.

1. Show that the coordinate rings of projectively equivalent varieties are isomorphic.
2. Show by example that the homogeneous coordinate ring of a projective variety is not invariant under isomorphisms, i.e. that there are isomorphic projective varieties X, Y such that the rings $k^{\mathbb{P}}[X]$ and $k^{\mathbb{P}}[Y]$ are not isomorphic.

Example 2.44 (Coordinate projection)**.** Let $[\mathbf{a}] = [1 : 0 : \cdots : 0]$ and consider the map

$$\pi : \mathbb{P}^n_k \setminus \{[\mathbf{a}]\} \to \mathbb{P}^{n-1}_k, \pi([b_0 : b_1 : \cdots : b_n]) = [b_1 : \cdots : b_n].$$

This is a regular map called the projection from $\mathbf{a}$ onto the hyperplane $V(X_0)$.

In Example 1.99 we considered the projection of a sphere from its north pole onto the horizontal coordinate plane. Recall that this was a partially defined rational map, undefined at the north pole. Below we project a projective "circle" C from its north pole and we find that the resulting projection is defined everywhere on C.

Example 2.45 (Stereographic projection in projective space)**.** Consider $C = V^{\mathbb{P}}(X^2 + Y^2 - Z^2) \subseteq \mathbb{P}^2_k$. In Example 2.10 we saw that $C \cap U_2$ is a circle. Consider the point $[\mathbf{a}] = [0 : 1 : 1] \in C$ and project from it onto the line $L = V(Y) \cong \mathbb{P}^1_k$. In coordinates, this map restricted to U_2 is

$$f([x : y : 1]) = \left[\frac{x}{y-1} : 0 : 1\right].$$

Homogenizing using the variable Z gives

$$f([X : Y : Z]) = \left[\frac{X}{Y-Z} : 0 : 1\right] = [X : 0 : Y - Z].$$

The above function is defined on $\mathbb{P}^2_k \setminus V^{\mathbb{P}}(-X, Y - Z) = \mathbb{P}^2_k \setminus \{[0 : 1 : 1]\} = \mathbb{P}^2_k \setminus \{\mathbf{a}\}$. But we can "massage" it to work at this point as well since

$$\begin{aligned} f([X : Y : Z]) &= [X : 0 : Y - Z] & \text{on } \mathbb{P}^2_k \setminus \{[0 : 1 : 1]\} \\ &= [X(Y + Z) : 0 : (Y - Z)(Y + Z)] \\ &= [X(Y + Z) : 0 : -X^2] \\ &= [Y + Z : 0 : -X] & \text{on } \mathbb{P}^2_k \setminus \{[0 : -1 : 1]\}. \end{aligned}$$

Then we can set $f : C \to L = \mathbb{P}^2_k$ to be given by

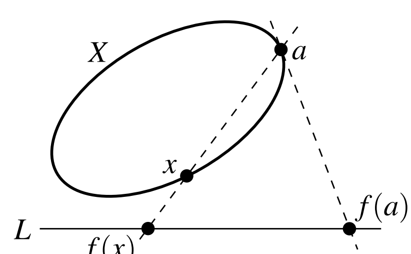
$$f([X : Y : Z]) = \begin{cases} [X : 0 : Y - Z]. & \text{on } C \setminus \{[0 : 1 : 1]\} \\ [Y + Z : 0 : -X] & \text{on } C \setminus \{[0 : -1 : 1]\} \end{cases}$$

and the computation above shows the two rules agree on the overlap. This shows that f is regular on C. In particular, with this rule

$$f(\mathbf{a}) = f([0 : 1 : 1]) = [2 : 0 : 0] = [1 : 0 : 0].$$

Intuitively, the "fix" here for defining $f([\mathbf{a}])$ is to say that the projection of $[\mathbf{a}]$ is obtained by taking the tangent line to C at $[\mathbf{a}]$ and intersecting this with L. Unlike in affine space where these two lines are parallel, in projective space these two lines intersect at the point at infinity of L, which is $[1 : 0 : 0]$.

See the figure below for an illustration.



(c)

Monday, February 17 & Wednesday, February 19

2.6 Interesting projective varieties

2.6.1 The Segre variety

So far we know that a product of affine spaces $\mathbb{A}_k^m \times \mathbb{A}_k^n = \mathbb{A}_k^{m+n}$ is an affine space, thus an affine variety. We don't know yet that a product of projective spaces $\mathbb{P}_k^m \times \mathbb{P}_k^n$ is an projective variety. The purpose of this section is to define a projective variety which is in bijection with $\mathbb{P}_k^m \times \mathbb{P}_k^n$, thus allowing us to identify $\mathbb{P}_k^m \times \mathbb{P}_k^n$ with a projective variety.

Definition 2.46. For integers $m \geq 0, n \geq 0$, the *Segre embedding* is the map

$$\Sigma_{m,n} : \mathbb{P}_k^m \times \mathbb{P}_k^n \to \mathbb{P}^{(m+1)(n+1)-1}$$

$$\Sigma_{m,n}([a_0 : \cdots : a_m], [b_0 : \cdots : b_n]) = [a_0b_0 : a_0b_1 : \cdots : a_0b_n : a_1b_0 : \cdots : a_1b_n : \cdots : a_mb_n].$$

The *Segre variety* $\mathfrak{S}_{m,n} = \mathrm{Im}(\sigma_{m,n}) \subseteq \mathbb{P}^{(m+1)(n+1)-1}$ is the image of the above map.

We will see below that the Segre map is a injection. We declare that the product $\mathbb{P}_k^m \times \mathbb{P}_k^n$ is a projective variety by identifying it with $\mathfrak{S}_{m,n}$ by means of the above map.

Example 2.47. Consider the map

$$\Sigma_{1,1} : \mathbb{P}_k^1 \times \mathbb{P}_k^1 \to \mathbb{P}_k^3, \varphi([s,t],[u,v]) = [su : sv : tu : tv].$$

We can see that this map is well defined since the identity

$$(su : sv : tu : tv) = (s,t) \cap (u,v)$$

ensures that

$$V^{\mathbb{A}}(su : sv : tu : tv) = V^{\mathbb{A}}(s,t) \cup V^{\mathbb{A}}(u,v) = \{(0,0)\} \times \mathbb{A}^2 \cup \mathbb{A}^2 \times \{(0,0)\}.$$

In words, this means that $su : sv : tu : tv$ vanish if and only if $s = t = 0$ or $u = v = 0$. But neither of these are possible since $[s : t], [u : v] \in \mathbb{P}_k^1$. Let $k[P_k^3] = k[Z_{0,0}, Z_{0,1}, Z_{1,0}, Z_{1,1}]$.

We see that this map is a injection by constructing a left inverse as follows:

$$\varphi : \mathfrak{S}_{1,1} \to \mathbb{P}_k^1 \times \mathbb{P}_k^1, \varphi([a_{0,0}, a_{0,1}, a_{1,0}, a_{1,1}]) = \begin{cases} [a_{0,0}, a_{0,1}] & \text{on } D(Z_{0,0}) \cap \mathfrak{S}_{1,1} \\ [a_{1,0}, a_{1,1}] & \text{on } D(Z_{1,0}) \cap \mathfrak{S}_{1,1}. \end{cases}$$

Note that $\varphi \circ \Sigma_{1,1}([s,t],[u,v]) = \begin{cases} [su : sv] = [u : v] & \text{if } s \neq 0 \\ [tu : tv] = [u : v] & \text{if } t \neq 0. \end{cases} = \mathrm{id}_{\Sigma_{1,1}}$ This also shows that φ is well defined (the two rules agree on the intersection of their domains).

Now let's show that $\mathfrak{S}_{1,1}$ is a projective variety.

Claim 2.48. $\mathfrak{S}_{1,1} = V^{\mathbb{P}}(Z_{0,0}Z_{1,1} - Z_{0,1}Z_{1,0})$. (See Exercise 2.11 for an illustration. Note that you can see two sets of parallel lines, representing the lines $\{pt\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{pt\}$.)

Consider the equation $F = Z_{0,0}Z_{1,1} - Z_{0,1}Z_{1,0}$. Then $F(su, sv, tu, tv) = 0$ implies $\mathfrak{S}_{1,1} \subseteq V^{\mathbb{P}}(Z_{0,0}Z_{1,1} - Z_{0,1}Z_{1,0})$. For the converse, take a point $[a_{0,0}, a_{0,1}, a_{1,0}, a_{1,1}] \in V^{\mathbb{P}}(Z_{0,0}Z_{1,1} - Z_{0,1}Z_{1,0})$. This means that the determinant of the matrix below is zero

$$\begin{bmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{bmatrix}.$$

Thus the rows are linearly dependent, that is, there exist $s, t, u, v \in k$ such that $(a_{0,0}\, a_{0,1}) = s(u\, v) = (su\, sv)$ and $(a_{1,0}\, a_{1,1}) = t(u\, v) = (tu\, tv)$. Thus $[a_{0,0}, a_{0,1}, a_{1,0}, a_{1,1}] \in \mathfrak{S}_{1,1}$.

We can do this more generally. Here is a better way of thinking about the map $\Sigma_{m,n}$. Regard a point in $\mathbb{P}^{(m+1)(n+1)-1}$ as a non-zero $(m+1) \times (n+1)$ matrix, with two such matrices deemed equal if one is a non-zero scalar times the other. Then we can think of the map $\Sigma_{m,n}$ as given by the (awkward) matrix product

$$\Sigma_{m,n}([\mathbf{a}], [\mathbf{b}]) = [\mathbf{a}^T \mathbf{b}]. \tag{2.5}$$

With this latter description (2.5), it is clear that the image of $\Sigma_{m,n}$ is contained in

$$W := \{[A] \in \mathbb{P}^{mn+m+n} \mid A \text{ is a } (m+1) \times (n+1) \text{ matrix of rank exactly one}\}$$

since the rank of a product of matrices is less or equal to the rank of each of the factors.

Theorem 2.49. *For any m and n, the Segre map is injective and the Segre variety $\mathfrak{S}_{m,n}$ is a projective subvariety of $\mathbb{P}^{(m+1)(n+1)-1}$. It can be described as the variety of matrices of rank one*

$$\begin{aligned} \mathfrak{S}_{m,n} &= V_k^{\mathbb{P}}\left(2 \times 2 \textit{ minors of } \begin{bmatrix} Z_{0,0} & Z_{0,1} & \cdots Z_{0,m} \\ Z_{1,0} & Z_{1,1} & \cdots Z_{1,m} \\ \vdots & \vdots & \cdots \vdots \\ Z_{n,0} & Z_{n,1} & \cdots Z_{n,m}. \end{bmatrix} \right) \\ &= V^{\mathbb{P}}(\{Z_{i,j}Z_{s,t} - Z_{i,t}Z_{s,j} \mid i < s, j < t\}). \end{aligned}$$

Proof. It is clear from (2.5) that $\Sigma_{m,n}$ is a well-defined function since $\mathbf{a}^T\mathbf{b} = \mathbf{0}$ if and only if $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$. By the remarks above we have that $\mathfrak{S}_{m,n} \subseteq W$. For the converse take a matrix $[A] \in W$. Then A has a non-zero row, call it $\mathbf{a}$ and so it can be written as

$$A = \begin{bmatrix} b_1\mathbf{a} \\ b_2\mathbf{a} \\ \vdots \\ b_m\mathbf{a} \end{bmatrix} = \mathbf{a}^T\mathbf{b}.$$

for some $b_1, \ldots, b_m$ not all zero (one of them is 1).

Write the $(m+1)(n+1)$ homogeneous coordinate functions of $\mathbb{P}^{(m+1)(n+1)-1}$ as $Z_{i,j}$ for $0 \leq i \leq m$ and $0 \leq j \leq n$. Since a matrix has rank one if and only if it is non-zero and all its 2×2 minors vanish, we have

$$\mathfrak{S}_{m,n} = W = V^{\mathbb{P}}(\{Z_{i,j}Z_{s,t} - Z_{i,t}Z_{s,j} \mid i < s, j < t\}).$$

This proves $\mathfrak{S}_{m,n}$ is a projective variety.

□

The Segre embedding leads to the notion of *Segre coordinates* for points on $\mathbb{P}^m \times \mathbb{P}^n$. The projective coordinate ring of $\mathfrak{S}_{m,n}$ is

$$k^{\mathbb{P}}[\mathfrak{S}_{m,n}] = k^{\mathbb{P}}[\mathbb{P}^{(m+1)(n+1)-1}]/I = k[Z_{i,j}]/I$$

where I is the ideal generated by the two-by-two determinants of the matrix $(Z_{i,j})_{i,j}$ of variables. The k-algebra map

$$k[Z_{i,j}] \to k[X_0, \ldots, X_m, Y_0, \ldots, Y_n], \quad Z_{i,j} \mapsto X_iY_j$$

induces and isomorphism

$$\theta : k^{\mathbb{P}}[\mathfrak{S}_{m,n}] \cong k[X_iY_j] \subseteq k[X_0, \ldots, X_m, Y_0, \ldots, Y_n] \tag{2.6}$$

The Segre embedding allows us to describe all closed subsets of $\mathbb{P}^m \times \mathbb{P}^n$.

Definition 2.50. Given two lists of variables $X_0, \ldots, X_m$ and $Y_0, \ldots, Y_n$, declare an element F of $k[X_0, \ldots, X_m; Y_0, \ldots, Y_n]$ to be *bihomogenous of bidegree* (d, e) if it is a k-linear combination of monomials of the form $X_0^{i_0} \cdots X_m^{i_m} Y_0^{j_0} \cdots Y_n^{j_n}$ such that $\sum_s i_s = d$ and $\sum_s j_s = e$.

Given such a bi-homogeneous polynomial F, we have

$$F(la_0, \ldots, la_m, \mu b_0, \ldots, \mu b_n) = l^d \mu^e F(a_0, \ldots, a_m, b_0, \ldots, b_n) \text{ for all } l, \mu \in k,$$

and in particular the condition on a point $([\mathbf{a}], [\mathbf{b}]) \in \mathbb{P}^m \times \mathbb{P}^n$ that $F([\mathbf{a}], [\mathbf{b}]) = 0$ is independent of representatives. We define

$$V^{\mathbb{P}^m_k \times \mathbb{P}^n_k}(F) = \{([\mathbf{a}], [\mathbf{b}]) \in \mathbb{P}^m \times \mathbb{P}^n \mid F(\mathbf{a}, \mathbf{b}) = 0\}.$$

If $G = G(\ldots, Z_{i,j}, \ldots)$ is a homogenous element in $k^{\mathbb{P}}[W]$ of degree d, then the map θ in (2.6) sends G to $F := \theta(G) = G(\ldots, X_iY_j, \ldots)$ in $k[X_0, \ldots, X_m; Y_0, \ldots, Y_n]$, which is bihomogeneous of bidegree (d, d). Moreover, it is clear from the formula for the isomorphism $\Sigma_{m,n} : \mathbb{P}^m \times \mathbb{P}^n \xrightarrow{\cong} \mathfrak{S}_{m,n}$ that it induces an isomorphism

$$\Sigma_{m,n} : V^{\mathbb{P}^m_k \times \mathbb{P}^n_k}(F) \cong V^{\mathbb{P}}_W(G).$$

Proposition 2.51. *Given a closed subset Z of $\mathbb{P}_k^m \times \mathbb{P}_k^n$, we have that*

$$Z = V^{\mathbb{P}_k^m \times \mathbb{P}_k^n}(F_1, \ldots, F_l)$$

for some bi-homogenous polynomials $F_1, \ldots, F_l$.

Proof. Let Z' be the closed subset of $\mathfrak{S}_{m,n}$ corresponding to Z under the isomorphism $\Sigma_{m,n}$ of Theorem 2.49. Then $Z' = V^{\mathbb{P}}_{\mathfrak{S}_{m,n}}(G_1, \ldots, G_l)$ where for each s, $G_s = G_s(Z_{i,j})$ is a homogenous element of $k^{\mathbb{P}}[\mathfrak{S}_{m,n}]$, say of degree d_s. As noted above, $V^{\mathbb{P}}_{\mathfrak{S}_{m,n}}(G_s)$ corresponds under $\Sigma_{m,n}$ to $V^{\mathbb{P}_k^m \times \mathbb{P}_k^n}(F_s)$ where $F_s = \theta(G_s) = G_s(X_iY_j) \in [X_0, \ldots, X_m; Y_0, \ldots, Y_n]$, which is bihomogeneous of bidegree (d_s, d_s). It follows that Z is cut out by the bihomogenous polynomials $F_1, \ldots, F_l$. □

Remark 2.52. There is a version of projective Nullstellensatz for $\mathbb{P}_k^m \times \mathbb{P}_k^n$. The *irrelevant ideal* of $\mathbb{P}_k^m \times \mathbb{P}_k^n$ is the ideal $B = (X_iY_j \mid 0 \leq i \leq m, 0 \leq j \leq n)$ that satisfies $V^{\mathbb{P}_k^m \times \mathbb{P}_k^n}(B) = \emptyset$.

2.6.2 The Veronese variety

Let $N = \binom{n+d}{d}$ denote the number of monomials of degree d in $k[X_0, \ldots, X_n]$. For any n and d, let $F_1, \ldots, F_N$ be a complete list of all the monomials of degree d in $X_0, \ldots, X_n$.

Definition 2.53. The d-th *Veronese subring* of $k[X_0, \ldots, X_n]$ is the ring

$$k[X_0, \ldots, X_n]^{(d)} = \{f \in k[X_0, \ldots, X_n] \mid \deg(f) \text{ is divisible by } d\}. \tag{2.7}$$

Thus $F_0, \ldots, F_N \in k[X_0, \ldots, X_n]^{(d)}$ and in fact they generate this ring as a k-algebra.

Example 2.54. If $n = 1$ then the list is $X^d, X^{d-1}Y, \ldots, Y^d$, so that $N = d$.

If $n = 2$ and $d = 4$, then we have X^4, X^3Y, X^2Y^2, XY^3, Y^4, X^3Z, X^2YZ, XY^2Z, Y^3Z, X^2Z^2, XYZ^2, Y^2Z^2, XZ^3, YZ^3, Z^4 and $N = 14$.

By (2.4) the function $\nu_d : \mathbb{P}^n \to \mathbb{P}^{N-1}$ given by

$$\nu_d([a_0 : \cdots : a_n]) = [F_1(a_0, \ldots, a_n) : \cdots : F_N(a_0, \ldots, a_n)]$$

is a regular map.

Definition 2.55. The *Veronese map* or *d-uple embedding* is the regular map

$$\nu_d : \mathbb{P}^n \to \mathbb{P}^{N-1} \quad \nu_d([a_0 : \cdots : a_n]) = \underbrace{[F_1(a_0, \ldots, a_n) : \cdots : F_N(a_0, \ldots, a_n)]}_{\text{all monomials of degree } d}. \tag{2.8}$$

The *Veronese variety* $V_{n,d} = \operatorname{Im}(\nu_d) \subseteq \mathbb{P}_k^{N-1}$ is the image of the above map.

It is not clear at this point that the Veronese variety is indeed a projective variety, but this is true and will be justified later (using the Closed Mapping Theorem). For the rest of the section we take this fact for granted.

Example 2.56. $\nu_{n,1}$ is the identity map and so $V_{n,1} = \mathbb{P}^n$.

Example 2.57. $V_{1,d} = \{[s^d : s^{d-1}t : s^{d-2}t^2 : \cdots : s^2t^{d-2} : st^{d-1} : t^d] \mid [s:t] \in \mathbb{P}^1_k\} \subseteq \mathbb{P}^d_k$ is called the *rational normal curve* of degree d.

Note that monomials in $n+1$ variables $X_0, \ldots, X_n$ are indexed by tuples $I = (i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$ where $\mathbb{N} = \{0, 1, \ldots\}$, where such an I corresponds to the monomial

$$X^I := X_0^{i_0} X_1^{i_1} \cdots X_n^{i_n}.$$

Define $|I| := \sum_s i_s$, so that the degree of X^I is $|I|$.

With this notation, the d-uple embedding

$$\nu_{n,d} : \mathbb{P}^n \to \mathbb{P}^{N-1}$$

is the map given by $[a_0 : \cdots : a_n] \mapsto [\cdots : X^I(a_0, \ldots, a_n) : \cdots]$ where I ranges over tuples $I \in \mathbb{N}^{n+1}$ with $|I| = d$. We should also specify an ordering on the I's to make $\nu_{n,d}$ a specific map — let's say we order them lexicographically, so that $(d, 0, \ldots, 0)$ comes first.

Let us write the coordinate functions of $\mathbb{P}^{N-1}$ as Z_I for $I \in \mathbb{N}^{n+1}$ with $|I| = d$, i.e.,

$$k^{\mathbb{P}}[\mathbb{P}^{N-1}] = k[\{Z_I \mid I \in \mathbb{N}^{n+1}, |I| = d\}].$$

Theorem 2.58. *The Veronese variety $V_{n,d}$ is isomorphic to $\mathbb{P}^n$ as projective varieties.*

Proof. The map $\nu_{n,d}$ is a well-defined regular map since the X^I's all have the same degree and $V^{\mathbb{P}}(X^I \mid |I| = d) = \emptyset$ since, e.g., $X_0^d, \ldots, X_n^d$, only vanish simultaneously at the origin in $\mathbb{A}^{n+1}$.

We will define a proposed inverse for $\nu_{n,d}$. Consider the open sets

$$U_i = D_{V_{n,d}}(Z_{\{0,\ldots,0,d,0,\ldots,0\}})$$

where the d is in the i-th spot and define

$$\varphi_{i,d} : U_i \to \mathbb{P}^n_k \qquad \varphi_{i,d}([a_I]) = [a_{\{1,0,\ldots,0,d-1,0,\ldots,0\}} : \cdots : a_{\{0,0,\ldots,0,d-1,0,\ldots,1\}}]$$

that is, $\varphi_{i,d}$ remembers only the coordinates indexed by tuples I where the i-th entry of I is $d-1$, exactly one other entry is 1 and all other entries are zero. We have

$$\varphi_{i,d} \circ \nu_{n,d}([x_0 : \cdots : x_n]) = [x_0 x_i^{d-1} : \cdots : x_n x_i^{d-1}] = [x_0 : \cdots : x_n] \text{ on } U_i$$

which also shows that $\varphi_{i,d} = \varphi_{j,d}$ on $U_i \cap U_j$, so there is a well-defined map

$$\varphi_d : V_d \to \mathbb{P}^n_k \quad \varphi_d = \varphi_{i,d} \text{ on } U_i$$

that is a left inverse to $\nu_{i,d}$. Since $\nu_{n.d}$ is surjective it follows that it is in fact bijective and φ_d is its inverse. We have shown that $V_{n,d}$ and $\mathbb{P}^n$ are isomorphic as projective varieties □

Consider the ideal of $k[\mathbb{P}^{N-1}]$

$$Q = \left(\{Z_I Z_J - Z_{I'} Z_{J'} \mid I, J, I', J' \in \mathbb{N}^{n+1}, |I| = |J| = |I'| = |J'| = d, I + J = I' + J'\}\right). \tag{2.9}$$

The formula for $\nu_{n,d}$ makes clear that its image is contained in $V^{\mathbb{P}}(Q)$. In fact we have that $V_{n,d} = V^{\mathbb{P}}(I)$.

Theorem 2.59. *The Veronese variety is defined by quadratic binomials as follows*

$$V_{n,d} = V^{\mathbb{P}}\left(\{Z_I Z_{I'} - Z_J Z_{J'} \mid I, J, I', J' \in \mathbb{N}^{n+1}, |I| = |J| = |I'| = |J'| = d, I + I' = J + J'\}\right).$$

The proof is omitted. A corollary of Theorem 2.58 gives a presentation of the Veronese ring.

Corollary 2.60. *There is a k-algebra isomorphism*

$$k[X_0, \ldots, X_n]^{(d)} \cong k[Z_I \mid |I| = d]/Q \qquad Z_I \mapsto X^I,$$

where Q is the ideal in (2.9).

Proof. The map $\nu_{n,d}$ in (2.8) induces a map $\nu_{n,d}^* : k[V_{n,d}] = k[Z_I \mid |I| = d]/Q \to k[\mathbb{P}^n_k]$ given by $\nu_{n,d}^*(Z_I) = X^I$. Since the monomials X^I with $|I| = d$ generate $k[X_0, \ldots, X_n]^{(d)}$ as a k-algebra, the image is $k[X_0, \ldots, X_n]^{(d)}$. Since the ideal of relations between monomials of degree d is $I^{\mathbb{P}}(V_n, d)$, the fact that the map is injective follows from Theorem 2.59. □

Example 2.61. The defining ideal of the rational normal curve in Example 2.57 is

$$I^{\mathbb{P}}(V_{1,d}) = \left(2 \times 2 \text{ minors of } \begin{bmatrix} Z_{0,d} & Z_{1,d-1} & \cdots & Z_{d-1,1} \\ Z_{1,d-1} & Z_{2,d-2} & \cdots & Z_{d,0} \end{bmatrix}\right).$$

Example 2.62. The defining ideal of the Veronese surface $V_{2,2} \subseteq \mathbb{P}^5_k$ is

$$I^{\mathbb{P}}(V_{2,2}) = \left(2 \times 2 \text{ minors of } \begin{bmatrix} Z_{0,0,2} & Z_{0,1,1} & Z_{0,2,0} \\ Z_{0,1,1} & Z_{1,0,1} & Z_{1,1,0} \\ Z_{0,2,0} & Z_{1,1,0} & Z_{2,0,0} \end{bmatrix}\right).$$

We will use the following later on:

Lemma 2.63. *Given a projective variety $Z \subseteq \mathbb{P}^n$, there is an integer d such that $Z = V^{\mathbb{P}}(F_1, \ldots, F_l)$ for some homogeneous polynomial F_i each of which has degree d.*

Given a projective variety $Z \subseteq \mathbb{P}^m \times \mathbb{P}^n$, there is an integer d such that

$$Z = V^{\mathbb{P} \times \mathbb{P}}(G_1, \ldots, G_l)$$

for some bihomogeneous polynomials G_i each of which has bidegree (d, d).

Proof. The first assertion is a homework problem.

For the second assertion, if we follow the proof of Proposition 2.51 starting with F_i's of degree d for some d (which is possible by what was just proved), then we arrive at a list of bi-homogeneous polynomials of bi-degree (d, d). □

One application of the Veronese embedding is to turn varieties of $\mathbb{P}^n_k$ cut out by degree d equations into varieties of $\mathbb{P}^{N-1}_k$ cut out by linear equations, that is, linear subspaces.

Lemma 2.64. *If $C \subseteq \mathbb{P}^n$ is a projective variety, then for some integer d we have*

$$C = \nu_{n,d}^{-1}(L)$$

where $\nu_{n,d} : \mathbb{P}^n \to \mathbb{P}^N$ is the d-uple embedding and $L \subseteq \mathbb{P}^{N-1}$ is a projective variety such that $I^{\mathbb{P}}(L)$ is generated by **linear** *homogeneous polynomials.*

Proof. Homework problem.

□

Friday, February 21

2.6.3 Grassmannian varieties

Recall that we defined projective space as the quotient

$$\mathbb{P}^n_k = \frac{\mathbb{A}^{n+1} \setminus \{0\}}{k^*}. \tag{2.10}$$

It can also be viewed as the set of lines through the origin in $\mathbb{A}^{n+1}$.

Grassmannians are generalizations of the projective space, where instead of lines through the origin we consider higher-dimensional linear spaces.

Definition 2.65. The *Grassmannian* $\mathrm{Gr}(d, n)$ is the set of d-dimensional subspaces of k^n.

Example 2.66. $\mathrm{Gr}(1, n+1) = \mathbb{P}^n_k$ since projective space can be identified with the set of all lines in $k^{n+1} = \mathbb{A}^{n+1}_k$.

An equivalent definition that parallels (2.10):

Definition 2.67. The *Grassmannian* $\mathrm{Gr}(d, n)$ is also defined as

$$\mathrm{Gr}(d, n) = \{A \in \mathcal{M}_{d,n}(k) \mid \mathrm{rank}(A) = d\}/\sim$$

where $A \sim B$ if and only if there exists $C \in GL_d(k)$ so that $A = CB$.
I will write $[A]$ for the equivalence class of A with respect to the equivalence relation $\sim$.

Th see that the two definitions are equivalent, notice that any matrix $A \in \mathcal{M}_{d,n}(k)$ with $\text{rank}(A) = d$ gives rise to a d-dimensional vector space of k^n known as the row space of A. Moreover, two matrices A, B have the same row space if and only if $A = CB$ for some $C \in GL_d(k)$, where C is the change of basis matrix converting between the rows of A and the rows of B as bases of the same row space.

Every element $[A] \in \text{Gr}(d, n)$ has a canonical representative, namely the reduced row echelon form of A. Thus $[A] = [RREF(A)]$ in $\text{Gr}(d, n)$. Notice for later that if A is $d \times n$ and it has d pivots then $\text{RREF}(A)$ has exactly $d(n-d)$ arbitrary entries in non-pivot columns and all the other entries in pivot columns are determined. For each list of indices $1 \le i_1 < i_2 < \ldots < i_d \le n$ we define a *distinguished set* of $\text{Gr}(d, n)$ by

$$U_{i_1,i_2,\ldots,i_d} = \{[A] \in \text{Gr}(d, n) \mid RREF(A) \text{ has pivots in columns } i_1, i_2, \ldots, i_d\}.$$

We now proceed to show that Grassmannians are projective varieties by constructing a map that embeds $\text{Gr}(d, n)$ in projective space and identifying $\text{Gr}(d, n)$ with its image through this map.

Definition 2.68. Given a matrix $A \in \mathcal{M}_{d,n}(k)$, write $\Delta_{i_1,\ldots,i_d}(A)$ for the determinant of the $d \times d$ submatrix formed by columns $i_1, \ldots, i_d$ of A for some list of indices $1 \le i_1 < i_2 < \ldots < i_d \le n$. Notice there are $\binom{n}{d}$ such tuples of indices.

The *Plücker map* $\Delta : \text{Gr}(d, n) \to \mathbb{P}^{\binom{n}{d}-1}$ is defined by

$$\Delta([A]) = \underbrace{[\Delta_{1,2,\ldots,d}(A) : \cdots : \Delta_{i_1,\ldots,i_d}(A) : \cdots : \Delta_{n-d+1,\ldots,n}(A)]}_{\text{all the } d\times d \text{ minors of } A}.$$

Observe that the Plücker map is well-defined: it does not yield $[0 : \cdots : 0]$ since for $[A] \in \text{Gr}(n, d)$ we have $\text{rank}(A) = d$ so at least one of the $d \times d$ minors of A is non zero. Moreover, if $A = CB$, then for all tuples of indices $1 \le i_1 < i_2 < \ldots < i_d \le n$ we have $\Delta_{i_1,\ldots,i_d}(A) = \det(C)\Delta_{i_1,\ldots,i_d}(B)$, which yields $\Delta([A]) = \Delta([B])$.

Example 2.69. Let $A = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$. The image of $[A]$ via the Plücker map is

$$\Delta([A]) = [1 : c : d : -a : -b : ad - bc].$$

It satisfies $\Delta_{12}(A)\Delta_{34}(A) - \Delta_{13}(A)\Delta_{24}(A) + \Delta_{14}(A)\Delta_{23}(A) = 0$. This is called a Plücker relation.

There is a more conceptual way to obtain the Plücker map, which we now define. First, using the standard basis $e_1, e_2, \ldots, e_n$ of k^n define a vector space

$$\bigwedge^d k^n = \text{Span}_k\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_d} \mid 1 \le i_1 < i_2 < \ldots < i_d \le n\}$$

where if u and v are in the same spot and the $\cdots$ are the same

$$\cdots \wedge u \wedge \cdots + \cdots \wedge v \wedge \cdots = \cdots \wedge (u + v) \wedge \cdots,$$

for any permutation σ on d elements we have

$$v_1 \wedge \cdots \wedge v_d = (-1)^{\mathrm{sgn}(\sigma)} v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(d)}$$

and for any $v \in k^n$

$$\cdots \wedge v \wedge \cdots \wedge v \wedge \cdots = 0.$$

Definition 2.70. Given a d-dimensional subspace V of k^n with basis $v_1, \ldots, v_d$ the *Plücker map* is given by

$$\Delta(L) = [v_1 \wedge \cdots \wedge v_d],$$

where $[v_1 \wedge \cdots \wedge v_d]$ denotes the vector of coefficients of $v_1 \wedge \cdots \wedge v_d$ with respect to the standard basis of $\bigwedge^d k^n$ consisting of $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_d}$ with $1 \leq i_1 < i_2 < \ldots < i_d$.

Example 2.71. Let $A = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$ be the same as in the previous example and let V be the row space of A. The image of V via the Plücker map is

$$\begin{aligned} \Delta(V) &= (e_1 + ae_3 + be_4) \wedge (e_2 + ce_3 + de_4) \\ &= e_1 \wedge e_2 + ce_1 \wedge e_3 + de_1 \wedge e_4 - ae_2 \wedge e_3 - be_2 \wedge e_4 + (ad - bc)e_3 \wedge e_4. \end{aligned}$$

With all this we have:

Theorem 2.72. *1. The image G of the Plücker map consists of those elements of $\bigwedge^d k^n$ that are totally decomposable, that is, can be written as $v_1 \wedge \cdots \wedge v_d$ for some $v_1, \ldots, v_d \in k^n$.*

2. The Plücker map is a bijection onto its image.

3. The image of $U_{i_1,\ldots,i_d}$ through the Plücker map is isomorphic to $\mathbb{A}^{d(n-d)}$.

4. The image G of the Plücker map is a projective variety with defining equations

$$G = V^{\mathbb{P}} \left(Z_{S\cup\{a,b\}} Z_{S\cup\{c,d\}} - Z_{S\cup\{a,c\}} Z_{S\cup\{b,d\}} + Z_{S\cup\{a,d\}} Z_{S\cup\{b,c\}} \mid 1 \leq a < b < c < d \leq n, \right.$$
$$\left. S \subseteq [n] \setminus \{a,b,c,d\}, |S| = d-2 \right).$$

Proof. I will just sketch (3) and partially (4).

For (3), fix an element $[A] \in U_{i_1,\ldots,i_d}$. We may assume A is in RREF and so it has $d(n-d)$ entries in non-pivot columns. Each of these entries is equal to one of the $d \times d$ minors of A, thus it appears in $\Delta([A])$. Ignoring the other entries of $\Delta([A])$ this gives a map to $\mathbb{A}_k^{d(n-d)}$. The inverse map takes an element of $\mathbb{A}_k^{d(n-d)}$ and inserts its coordinates into an RREF matrix to produce an element of $\mathrm{Gr}(d,n)$.

For (4), I will show that G is a projective variety but not necessarily that it has the claimed defining equations. For $w \in \bigwedge^d k^n$ consider the linear map $f_w : k^n \to \bigwedge^{d+1} k^n$, $f_w(v) = w \wedge v$. Then w is in G if and only if w is totally decomposable as $w = v_1 \wedge \cdots \wedge v_d$ if and only if $\mathrm{Ker}(f_w) = \mathrm{Span}_k\{v_1, \ldots, v_d\}$ is d-dimensional if and only if the rank of f_w is $n-d$ if and only if all $(n-d+1)\times(n-d+1)$ minors of the matrix representing the map f_w are equal to zero. These are polynomial equations in the coefficients of w in terms of the standard basis of $\bigwedge^d k^n$ (see the following example), and G is determined by their vanishing. □

Example 2.73. Consider $d=2, n=4$. We will set up the map f_w in the previous proof. Let

$$w = \Delta_{12}e_1\wedge e_2 + \Delta_{13}e_1\wedge e_3 + \Delta_{14}e_1\wedge e_4 + \Delta_{23}e_2\wedge e_3 + \Delta_{24}e_2\wedge e_4 + \Delta_{34}e_3\wedge e_4.$$

Listing $f_w(e_1), f_w(e_2), f_w(e_3)$, and $f_w(e_4)$ as columns vectors written in the basis $e_1\wedge e_2\wedge e_3, e_1\wedge e_2\wedge e_4, e_1\wedge e_3\wedge e_4, e_2\wedge e_3\wedge e_4$, the matrix of f_w is

$$\begin{bmatrix} \Delta_{23} & -\Delta_{13} & \Delta_{12} & 0 \\ \Delta_{24} & -\Delta_{14} & 0 & \Delta_{12} \\ \Delta_{34} & 0 & -\Delta_{14} & \Delta_{13} \\ 0 & \Delta_{34} & -\Delta_{24} & \Delta_{23} \end{bmatrix}$$

and the proof above shows that $\mathrm{Gr}(2,4)$ is cut out by the 3×3 minors of the above matrix which are all of the form

$$\pm\Delta_{ij}(\Delta_{12}\Delta_{34} - \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23}).$$

So we see that

$$\mathrm{Gr}(2,4) = \mathbb{V}^{\mathbb{P}}(Z_{ij}(Z_{12}Z_{34}-Z_{13}Z_{24}+Z_{14}Z_{23}) \mid 1\le i<j\le 4) = V^{\mathbb{P}}(Z_{12}Z_{34}-Z_{13}Z_{24}+Z_{14}Z_{23}).$$

Monday, February 24

2.7 Quasi-projective varieties

We have already discussed affine varieties and projective varieties. We now generalize what we mean by a "variety".

Definition 2.74. Define a

- *classical affine variety* to be a Zariski closed set $X = V^{\mathbb{A}}(I) \subseteq \mathbb{A}^n_k$ for some ideal I of $k[x_1,\dots,x_n]$;
- *projective variety* to be a Zariski closed set $X = V^{\mathbb{P}}(I) \subseteq \mathbb{P}^n_k$ for some homogeneous ideal I of $k[X_0,\dots,X_n]$;
- *quasi-affine variety* to be a Zariski open set U of a classical affine variety $X\subseteq \mathbb{A}^n_k$;
- *quasi-projective variety* to be a Zariski open set U of a projective variety $X\subseteq \mathbb{P}^n_k$.

Definition 2.75. For a quasi-projective variety X, a *regular map* $f: X\to \mathbb{P}^m_k$ is defined in the same was as for projective varieties; see Definition 2.35.

For X, Y quasi-projective varieties, a *regular map* or *morphism* $f: X\to Y$ is a regular map $f: X\to\mathbb{P}^m_k$ whose image lands in Y. An *isomorphism* between X and Y is a regular map which has a regular map for its inverse. If an isomorphism between X and Y exists, then X and Y are called *isomorphic*.

From now on we re-define what we means that a set X is a certain type of variety (affine/projective/quasi-affine) to mean that X is a quasi-projective variety which is isomorphic to an affine/projective/quasi-affine variety Y.

Example 2.76. $\mathbb{A}^n$ is affine, quasi-afine, quasi-projective, but not projective. The first two are clear. It is quasi-projective because $A^n_k \cong U_0 \subseteq \mathbb{P}^n_k$. To see that it's not projective see Example 2.98.

Example 2.77. $k^* = \mathbb{A}^1_k \setminus \{0\}$ is affine, quasi-affine, and quasi-projective, but it is not projective (we'll see this later). It's quasi-affine because it's open in $\mathbb{A}^1_k$. It is quasi-projective because $\mathbb{A}^1_k \setminus \{0\} \cong \mathbb{P}^1 \setminus \{0, \infty\}$. It is affine because it is isomorphic to $V(xy-1) \subseteq \mathbb{A}^2_k$.

Exercise 2.78. Prove that all types of varieties, i.e., classical affine/projective/quasi-affine are quasi-projective.

An important result.

Theorem 2.79 (Quasi-projective varieties are locally affine)**.** *If $X \subseteq \mathbb{P}^n_k$ is a quasi-projective variety, for every $\mathbf{a} \in X$ there is an open sub-set U of X such that $\mathbf{a} \in U$ and U is isomorphic to a classic affine variety.*

Proof. Without loss of generality, we may assume $\mathbf{a} \in X_0 = X \cap U_0 \cong X \cap \mathbb{A}^n_k$. Then X_0 is Zariski open in $\overline{X_0} \subseteq \mathbb{A}^n_k$. If $X_0 = \overline{X_0}$ then we are done by setting $U = X_0$. So assume $\emptyset \neq \overline{X_0} \setminus X_0$ and pick $0 \neq f \in k[X_0]$ so that $f(\mathbf{a}) \neq 0$ and set $U = D_{\overline{X_0}}(f)$. Observe that $\mathbf{a} \in U$ and U is open in X_0, hence also in X.

In $\mathbb{A}^{n+1}_k$ with coordinate ring $k[\mathbb{A}^{n+1}_k] = k[t_1, \ldots, t_{n+1}]$, set

$$H = V^{\mathbb{A}}(t_{n+1}f - 1) \text{ and } Y = (\overline{X_0} \times \mathbb{A}^1_k) \cap H.$$

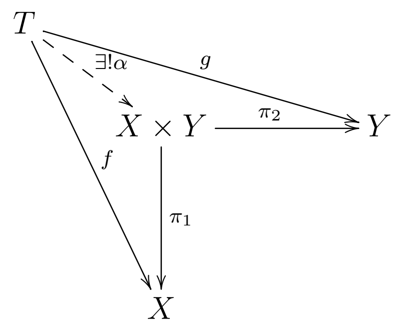
Observe that Y is a classical affine variety. We'll show that $U \cong Y$. Indeed, consider the maps $g : U \to Y, f(x) = (x, 1/f(x))$ and $h : Y \to U, h(x, t_{n+1}) = x$. They are clearly inverse maps and they are regular maps of quasi-projective varieties. □

The upshot of Theorem 2.79 is twofold:

- whenever we want to check a property of a quasi-projective variety X that is local, we can assume that X is a classical affine variety
- a regular function on a quasi-projective variety X can be thought of an open neighborhood of $\mathbf{a}$ as an element of $\mathcal{O}_{\overline{X_0}}(D(f)) = k[\overline{X_0}][1/f]$ for X_0, f as in the proof of the Theorem.

2.7.1 Products of quasi-projective varieties

Definition 2.80. If X and Y are varieties, their *product*, if it exists, is a variety $X \times Y$ equipped with morphisms $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ that is "universal" among all such diagrams — i.e., if T is any variety and $f : T \to X$ and $g : T \to Y$ are morphisms, there exists a unique morphism $\alpha : T \to X \times Y$ causing the evident two triangles to commute: $\pi_1 \circ \alpha = f$ and $\pi_2 \circ \alpha = g$.



Example 2.81. For instance, suppose $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ are two classical affine varieties. Then we may form the subset $X \times Y$ of $\mathbb{A}^{m+n}$ by identifying $\mathbb{A}^{m+n}$ with $\mathbb{A}^m \times \mathbb{A}^n$ in the evident way. Moreover, if $k[X] = k[x_1, \dots, x_m]/I$ and $k[Y] = k[y_1, \dots, y_n]/J$ for radical ideals I and J, then

$$k[X \times Y] = k[x_1, \dots, x_m, y_1, \dots, y_n]/L$$

where L is the smallest ideal of $k[x_1, \dots, x_m, y_1, \dots, y_n]$ containing I and J. (The hardest part in proving this is that L is radical.) Another way of stating this is that $k[X \times Y] \cong k[X] \otimes_k k[Y]$ (see the homework).

Theorem 2.82. *Every pair of quasi-projective varieties has a product.*

Proof. We will just sketch the construction, and we won't verify it works: suppose $X \subseteq \mathbb{P}_k^m$ and $Y \subseteq \mathbb{P}_k^n$ are quasi-projective varieties. Recall that $\Sigma_{m,n} : X \times Y \to P^N$ where $N = (m+1)(n+1) - 1$ denotes the Segre map and that this map is injective. Then $\Sigma_{m,n} : X \times Y \to \Sigma_{m,n}(X \times Y)$ is a bijection and one can show that $\Sigma_{m,n}(X \times Y)$ is a quasi-projective variety. Identifying $X \times Y$ with $\Sigma_{m,n}(X \times Y)$ yields that the former is quasi-projective.

□

Wednesday, February 26

2.8 The closed mapping theorem

2.8.1 Separated Varieties.

To motivate the following definition, recall that a topological space T is Hausdorff if and only if the subset $\Delta(T) = \{(t,t) \mid t \in T\}$ of the product space $T \times T$, equipped with the product topology, is a closed subset.

Definition 2.83. A variety X is *separated* if $\Delta(X) = \{(x,x) \mid x \in X\}$ is closed subset of the product variety $X \times X$ with respect to the Zariski topology.

Example 2.84. Set $k[\mathbb{A}^{2n}] = k[x_1, \ldots, x_n y_1, \ldots, y_n]$. For any positive integer n, $\mathbb{A}^n_k$ is separated since its diaginal is a closed set according to the description

$$\Delta(\mathbb{A}^n_k) = \{(\mathbf{a}, \mathbf{b}) \mid \mathbf{a}, \mathbf{b} \in \mathbb{A}^n, \mathbf{a} = \mathbf{b}\} = V^{\mathbb{A}}(x_1 - y_1, \ldots, x_n - y_n).$$

Example 2.85. Set $k[\mathbb{P}^n \times \mathbb{P}^n] = k[X_0, \ldots, X_n, Y_0, \ldots, Y_n]$. The diagonal is given by

$$\Delta(\mathbb{P}^n_k) = \{([\mathbf{a}], [\mathbf{b}]) \mid [\mathbf{a}], [\mathbf{b}] \in \mathbb{P}^n, [\mathbf{a}] = [\mathbf{b}]\} = V^{\mathbb{P}}\left(2 \times 2 \text{ minors of } \begin{bmatrix} X_0 & X_1 & \cdots & X_n \\ Y_0 & Y_1 & \cdots & Y_n \end{bmatrix}\right).$$

Indeed, a pair of points $[a_0 : \cdots : a_n]$ and $[b_0 : \cdots : b_n]$ in $\mathbb{P}^n$ are the same if and only if the $2 \times (n+1)$ matrix they determine has rank one. Note that the 2×2 minors of the above matrix are bihomogeneous of bidegree $(1, 1)$.

An alternative proof: Under the isomorphism $\mathbb{P}^n \times \mathbb{P}^n \cong W$, where $W \subseteq \mathbb{P}^{n^2+2n}$ is the Segre variety denotes $\mathfrak{S}_{n,n}$ in Definition 2.46. The diagonal subset $\Delta(\mathbb{P}^n)$ corresponds with the set cut out by the collection of homogenous polynomials $Z_{i,j} - Z_{j,i}$ in $k[\mathbb{P}^{n^2+2n}] = k[Z_{i,j} \mid 1 \le i, j \le n]$, and hence is closed in W.

We can generalize these examples as follows.

Proposition 2.86. *(1) All affine varieties are separated.*
(2) Any open or closed subset of a separated variety is again a separated variety.
(3) All quasi-affine varieties are separated.
(4) All quasi-projective varieties are separated.

Proof. (1) Set $k[\mathbb{A}^{2n}] = k[x_1, \ldots, x_n y_1, \ldots, y_n]$. If $X \subseteq \mathbb{A}^n$ is a classical affine variety, say

$$X = V^{\mathbb{A}}(f_1(x_1, \cdots, x_n), \ldots, f_p(x_1, \ldots, x_n)),$$

then $X \times X \subseteq \mathbb{A}^{2n}$ is the closed set given by

$$X \times X = \mathbb{V}^{\mathbb{A}}(f_1(x_1, \ldots, x_n), \ldots, f_p(x_1, \ldots, x_n), f_1(y_1, \ldots, y_n), \ldots, f_p(y_1, \ldots, y_n),$$

Since

$$\Delta(X) = (X \times X) \cap \ \Delta(\mathbb{A}^n_k)$$

and we have seem in Example 2.83 that $\Delta(\mathbb{A}^n_k)$ is closed, the conclusion follows.

(2) If Y is an open or closed subset of a separated variety X, then $\Delta(Y) = Y \times Y \cap \Delta(X)$, which is closed in $Y \times Y$ since $\Delta(X)$ is closed in $X \times X$.

(3) and (4) Follow from (2) and Example 2.83 or Example 2.84, respectively. □

So, every variety we care deeply about is separated. But non-separated spaces do exist:

Example 2.87 (Line with a double point)**.** Let X be the topological space obtained from gluing one copy of $\mathbb{A}^1$ to another copy of $\mathbb{A}^1$ by identifying the open subset U_0 of the first with the open subset U_0 of the second using the identity map. This space X is not separated.

2.8.2 The graph of a morphism.

Suppose $f : X \to Y$ is a morphism of quasi-projective. varieties. By the universal mapping property of products, we obtain a morphism of quasi-projective varieties

$$j : X \to X \times Y, \quad j = (\mathrm{id}_X, f) \text{ i.e., } j(x) = (x, f(x)).$$

such that $\pi_1 \circ j = \mathrm{id}_X$ and $\pi_2 \circ j = f$.

Definition 2.88. The image of j is known as the *graph* of f and is written as Γ_f

$$\Gamma_f = \{(x, f(x) \mid x \in X\}.$$

Proposition 2.89. *If Y is separated, then Γ_f is a closed subvariety of $X \times Y$ and the morphism j induces an isomorphism $X \cong \Gamma_f$ and f factors as $X \xrightarrow{\cong} \Gamma_f \xrightarrow{\pi_2} Y$.*

Proof. We show $\mathrm{Im}(j)$ is closed. Consider the morphism

$$g = f \times \mathrm{id}_Y : X \times Y \to Y \times Y$$

and observe that $\Gamma_f = j(X) = g^{-1}(\Delta(Y))$. Since we assume Y is separated, $\Delta(Y)$ is closed and hence $g^{-1}(\Delta(Y)) = j(X) = \Gamma_f$ is closed. The morphism j is one-to-one since π_1 is a left inverse of it. It follows $j : X \xrightarrow{\cong} \Gamma_f$ is an isomorphism, since the inverse π_1 is a morphism. □

Remark 2.90. When Y is not necessarily separated, it can be shown that the graph of f is an open subset of a closed subset of $X \times Y$, and hence is still a quasi-projective variety.

Exercise 2.91. Show that the assumption that Y is separated in Proposition 2.89 is necessary.

2.8.3 The Closed Mapping Theorem

In this section we prove that various maps are closed. Recall that a continuous function $f : X \to Y$ between two topological spaces X and Y is called *closed* if $f(C)$ is closed in Y for every closed subset C of X.

Theorem 2.92 (Closed Mapping Theorem)**.** *If $X \subseteq \mathbb{P}^m_k$ is a projective variety, Y is quasi-projective, and $f : X \to Y$ is a regular map, then f is closed. In particular,* $\mathrm{Im}(f)$ *is a closed subset of Y.*

This helps show that various sets we have encountered are projective varieties.

Example 2.93. For each n and d, the Veronese variety $V_{n,d}$ in Definition 2.55 is a projective variety because it is the image of a regular map $\nu_{n,d} : \mathbb{P}^n_k \to \mathbb{P}^{N-1}$ in (2.8).

Recall from Proposition 2.89 that a regular map $f : X \to Y$ factors as $X \xrightarrow{\cong} \Gamma_f \xrightarrow{\pi_2} Y$. Since homeomorphisms are closed, and compositions of closed maps are closed, to prove that f is closed it suffices to prove that the projection $\pi_2 : X \times Y \to Y$ is closed.

Theorem 2.94 (Projective varieties are complete)**.** *If $X \subseteq \mathbb{P}_k^m$ is a projective variety and Y is quasi-projective variety then the regular map $\pi_2 : X \times Y \to Y$ is closed.*

We will first show this for the special case $X = \mathbb{P}_k^m, Y = \mathbb{P}_k^n$.

Friday, February 28

Lemma 2.95. *For all m and n, the projection map $\pi_2 : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^n$ is closed (for the Zariski topologies).*

Proof. Let Z be any closed subset of $\mathbb{P}^m \times \mathbb{P}^n$. Then $Z = V^{\mathbb{P}\times\mathbb{P}}(F_1, \dots, F_l)$ where each $F_i(\vec{X}, \vec{Y})$ is bi-homogenous element of $k[\mathbb{P}^m \times \mathbb{P}^n] = k[X_0, \dots, X_m; Y_0, \dots, Y_n]$. Then

$$\begin{aligned}\pi_2(Z) &= \{[\mathbf{b}] \in \mathbb{P}^n \mid \exists [\mathbf{a}] \in \mathbb{P}_k^m \text{ such that } F_1(\mathbf{a}, \mathbf{b}) = \cdots = F_l(\mathbf{a}, \mathbf{b}) = 0\} \\ &= \left\{[\mathbf{b}] \in \mathbb{P}^n \mid \emptyset \neq V^{\mathbb{P}}(F_1(X_0, \dots, X_m, \mathbf{b}), \dots, F_l(X_0, \dots, X_m, \mathbf{b})) \subseteq \mathbb{P}^m\right\}.\end{aligned}$$

We need to show $\pi_2(Z)$ is the common zero locus of some homogenous polynomials.

Denote $\vec{Y} = Y_0, \dots, Y_n$. To gain some intuition, let us first prove this when each F_i is bi-homogenous of bi-degree $(1, d_i)$ for some d_i — that is

$$F_i = G_{i,0}(\vec{Y})X_0 + G_{i,1}(\vec{Y})X_1 + \cdots + G_{i,m}(\vec{Y})X_m$$

for some $G_{i,j}(\vec{Y})$'s that are homogenous of degree d_i. Let $G(\vec{Y})$ be the $(n+1) \times (m+1)$ matrix of polynomials $(G_{i,j}(\vec{Y}))_{i,j}$, and for a point $[\mathbf{b}] \in \mathbb{P}^n$, let $G([\mathbf{b}])$ be the $(n+1) \times (m+1)$ matrix $G([\mathbf{b}]) := (G_{i,j}(\mathbf{b}))_{i,j}$. (This matrix is only well-defined up to a non-zero scalar, but this won't matter in the proof.) Then $[\mathbf{b}] \in \pi_2(Z)$ if and only if the matrix equation

$$G([\mathbf{b}]) \cdot \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_m \end{bmatrix} = \vec{0}$$

has a non-zero solution iff the matrix $G(\mathbf{b})$ has rank strictly less than $m+1$ iff the $(m+1) \times (m+1)$ minors of $G(\mathbf{b})$ are all 0. (Note that each of these conditions is unchanged upon scaling a matrix by a non-zero scalar.) In other words,

$$\pi_2(Z) = V^{\mathbb{P}}\left(\text{the } (m+1) \times (m+1) \text{ minors of } G(\vec{Y})\right).$$

The minors are homogeneous polynomials, since for each row of $G(\vec{Y})$ each entry has the same degree. This proves $\pi_2(Z)$ is closed in $\mathbb{P}_k^n$ in our special case.

For the general case, by Lemma 2.63, we may assume each F_i is bi-homogenous of the same bidegree (d,e) for a fixed d and e. (We can even arrange so that $d=e$.)

Denote $F_i(X_0,\ldots,X_m,\mathbf{b}) = f_i(\vec{X})$. We have

$$\begin{aligned}
& \emptyset = V^{\mathbb{P}}(f_1(\vec{X}),\ldots,f_l(\vec{X})) \\
\iff\quad & k[X_0,\ldots,X_m]_{d'} \subseteq (f_1(\vec{X}),\ldots,f_l(\vec{X})) \text{ for some } d' \geq d \text{ by Prop. 2.24(3)} \\
\iff\quad & k[X_0,\ldots,X_m]_{d'} = (f_1(\vec{X}),\ldots,f_l(\vec{X}))_{d'} \\
\iff\quad & F_{d'} : (k[X_0,\ldots,X_m]_{d-d'})^l \to k[X_0,\ldots,X_m]_{d'}, F_{d'}(h_1,\ldots h_l) = \textstyle\sum_{i=1}^l h_i f_i
\end{aligned}$$

is surrjective. Denote $M_{d'}$ the matrix of the linear transformation $F_{d'}$ with respect to some chosen bases. The entries of $M_{d'}$ are the coefficients of the polynomials f_i, which are polynomials in the entries of $\mathbf{b}$. We write $M_{d'}(\vec{Y})$ for the matrix $M_{d'}$ with b_i replaced by Y_i. Then

$$F_{d'} \text{ is surjective } \iff \operatorname{rank}(M_{d'}) = \binom{m+d'}{d'}.$$

Taking contrapositive of all the work above we have proven

$$\begin{aligned}
\pi_2(Z) = \quad & \left\{[\mathbf{b}] \in \mathbb{P}_k^m \mid \emptyset \neq V^{\mathbb{P}}(f_1(\vec{X}),\ldots,f_l(\vec{X}))\right\} \\
= \quad & \textstyle\bigcap_{d'\geq d} V^{\mathbb{P}}\left(\binom{m+d'}{d'} \times \binom{m+d'}{d'} \text{ minors of } M_{d'}(\vec{Y})\right).
\end{aligned}$$

The set above is closed as it is an intersection of closed sets. □

We now return to Theorem 2.94.

Proof of Theorem 2.94. We first show the case $X = \mathbb{P}^m$ and Y arbitrary.

If $Y = \mathbb{P}^n$ then this is the content of Lemma 2.95.

If Y is a projective variety (i.e., a closed subset of $\mathbb{P}^n$ for some n), then this holds since $\mathbb{P}^m \times Y$ is closed in $\mathbb{P}^m \times \mathbb{P}^n$ and $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^n$ is closed.

Suppose Y is an affine variety. Then there is a projective variety Z such that Y is an open subset of Z. (If $Y \subseteq \mathbb{A}^n$ is a classical affine variety, then we may take Z to be the closure of Y in $\mathbb{P}^n$ where we regard $\mathbb{A}^n$ as an open subset of $\mathbb{P}^n$ in any of the standard ways.) Then $\pi_2 : \mathbb{P}^m \times Z \to Z$ is a closed mapping by what we have already proven. For any closed subset C of $\mathbb{P}^m \times Y$, let $\overline{C}$ be its closure in $\mathbb{P}^m \times Z$. Since $\overline{C} \cap (\mathbb{P}^m \times Y) = C$ it follows that

$$\pi_2(\overline{C}) \cap Y = \pi_2(\overline{C} \cap (\mathbb{P}^m \times Y)) = \pi_2(C),$$

and hence that $\pi_2(C)$ is closed in Y.

Finally let Y be a quasi-projective variety. Then $Y = U_1 \cup \cdots \cup U_m$ for open subvarieties U_i of Y. Recall the property of a subset S of a topological space T being closed is a "local property"; i.e., if $T = \bigcup U_j$ for open subsets U_j of T and $S \cap U_j$ is closed in U_j for all j, then S is closed in T. With that in mind, given a closed subset C of $\mathbb{P}^m \times Y$, we have that $\pi_2(C)$ is closed in Y iff $\pi_2(C) \cap U_i$ is closed in U_i for all i iff $\pi_2(C \cap (\mathbb{P}^m \times U_i))$ is closed in U_i for all i, and the latter holds since U_i is affine.

We have proven the clain for $X = \mathbb{P}^m$. Finally, if X is any projective variety, then X is a closed subvariety of $\mathbb{P}^m$ and so the claim follows since $X \times Y$ is a closed sub-variety of $\mathbb{P}^m \times Y$, and hence $C \times Y$ is a closed sub-variety of $\mathbb{P}^m \times Y$ for any closed subset C of X. □

The Closed Mapping Theorem has consequences on regular functions.

Corollary 2.96. *If X is a connected projective variety, then $\mathcal{O}_X(X) = k$.*

Proof. The assertion is equivalent to the statement that every morphism of varieties of the form $f : X \to \mathbb{A}^1_k$ is constant. View $\mathbb{A}^1$ as an open sub-variety of $\mathbb{P}^1$ so that we may regard f as a morphism from X to $\mathbb{P}^1_k$. Since $\mathbb{P}^1_k$ is separated and X is complete, $f(X)$ is closed in $\mathbb{P}^1_k$ by Theorem 2.92. But it is not equal to all of $\mathbb{P}^1_k$ and so $f(X)$ must be a finite set of points. Since X is connected, $f(X)$ must be a single point. (If $f(X)$ contained two points P and Q, then $f^{-1}(Q)$ and $f^{-1}(P)$ would be disjoint, open-and-closed subsets of X, contrary to X being connected.) □

2.8.4 Complete varieties

Just as with the notion of "separated", the concept of "complete" is motivated by a topological one. Also "recall" that:

> A topological space X is compact if and only if for all topological spaces Y the projection map $\pi : X \times Y \to Y$ is a closed mapping, where $X \times Y$ is endowed with the product topology.

I'll also remind you that a compact subset of a Hausdorff space is closed, and from this one deduces that if $f : X \to Y$ is continuous, X is compact, and Y is Hausdorff, then $f(X)$ is closed in Y.

Definition 2.97. A variety X is called *complete* (or sometimes "proper") if it is separated and for all varieties Y the morphism of varieties $\pi : X \times Y \to Y$ given by projection is a closed mapping (for the Zariski topologies).

Example 2.98. $X = \mathbb{A}^1$ is not complete: Take $Y = \mathbb{A}^1$ and $C = V(xy-1) \subseteq X \times Y = \mathbb{A}^2$. Then C is closed but $\pi(C) = \{a \in \mathbb{A}^1 \mid a \neq 0\}$ is not.

This proves that $\mathbb{A}^1$ is not projective, as projective varieties are complete.

Exercise 2.99. Prove that a closed sub-variety Z of a complete variety X is complete.

To illustrate the value of this notion, let me state and prove a generalization of Theorem 2.92:

Proposition 2.100 (Closed Mapping Theorem – General Version). *If X is a complete variety, then for every morphism $f : X \to Y$ of varieties such that Y is separated, the image of f in Y is a closed sub-variety.*

Proof. Recall that f factors as
$$X \xrightarrow{\cong} \Gamma_f \xrightarrow{\pi_2} Y$$
and, since we assume Y is separated, Γ_f is a closed sub-variety of $X \times Y$. Since we assume X is complete, $\pi_2(\Gamma_f) = \mathrm{Im}(f)$ is a closed subset of Y. The second assertion holds since closed sub-varieties of complete varieties are complete. □

Chapter 3

Dimension Theory

Monday, March 3

3.1 Dimension of affine varieties

Definition 3.1. The *dimension* of a noetherian topological space X, is

$$\dim(X) = \sup\{n \mid \emptyset \subsetneq Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n \subseteq X,\ Y_i \text{ irreducible closed in } X\}.$$

The dimension of the empty set is $-\infty$. It is possible that the dimension of a noetherian space is infinite.

Definition 3.2. Given a closed irreducible subset Y of X, the *codimension* of Y in X, written $\operatorname{codim}_X(Y)$ is he supremum over all n such that a choice of the form

$$\operatorname{codim}_X(Y) = \sup\{n \mid Y = Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n \subseteq X,\ Y_i \text{ irreducible closed in } X\}.$$

When Y is reducible, one defines $\operatorname{codim}_X(Y)$ to be the *minimum* value of $\operatorname{codim}_X(Y')$ where Y' ranges over the irreducible components of Y.

Remark 3.3. Topologize $\mathbb{N}$ by declaring a subset to be closed if and only if it is of the form $\{0, 1, \dots, n\}$ for some $-1 \leq n \leq \infty$. Then $\mathbb{N}$ is a noetherian space but has infinite dimension. There exist noetherian commutative rings R such that $\dim \operatorname{Spec}(R) = \infty$. (This cannot occur for local rings nor for rings that are finitely generated as k-algebras.)

Remark 3.4. Observe that every maximal chain of irreducible closed subsets (i.e., every chain that cannot be made longer and inserting new terms) of X has Y_n equal to an irreducible component of X and has Y_0 equal to a single point. In particular,

$$\dim(X) = \max\{\dim(Y) \mid Y \text{ is an irreducible component of } X\}$$

and

$$\dim(X) = \max\{\operatorname{codim}_X(P) \mid P \text{ is any point of } X\}.$$

Example 3.5. Let $X = V(xy, xz)$, the union of the x-axis and the yz-plane. The dimension of X is two. It is clearly at least two, since we have the chain with Y_0 being any point on the yz-plane, Y_1 being any line in this plane and containing this point, and Y_2 being the whole plane. It is harder to see that it is at most two. This is an example of a variety that fails to be "equi-dimensional" — the variety has two irreducible components, but they have different dimensions.

Let $Y = V(x) \subseteq X$, the yz-plane. Then $\operatorname{codim}_X(Y) = 0$ since the only closed subsets strictly between Y and X are given by Y together with a finite set of points on the line.

Let $Z = V(y, z)$, the x-axis. Then $\operatorname{codim}_X(Z)$ is also 0 by similar reasoning. Since $\dim(Z) = 1$, this is a bit counter-intuitive.

Let $W = V(x, y, z)$. When $\operatorname{codim}_X(W) = 2$. It's at least two since we have the chain $W \subsetneq V(x, y) \subsetneq Y \subseteq X$; it is less obvious, but true, that it is exactly two.

The following result follows from the fact that there is an order-reversing bijections between primes ideals of $k[X]$ and irreducible closed subsets of X:

Proposition 3.6. *The dimension of an affine variety X is equal to the Krull dimension of its coordinate ring $k[X]$, in particular $\dim(X)$ is finite. The codimension of an irreducible closed subset Y of X is the height of the corresponding prime ideal $I(Y)$ of $k[X]$ and the codimension of an arbitrary closed subset Y is the height of the corresponding radical ideal $I(Y)$.*

Since the Krull dimension of $k[x_1, \ldots, x_n] = n$ we have:

Corollary 3.7. *The dimension of $\mathbb{A}^n_k$ is n.*

Theorem 3.8 (Properties of dimension)**.** *Assume k is algebraically closed and X and Y are irreducible affine k-varieties*

1. *We have* $\dim(X \times Y) = \dim(X) + \dim(Y)$.
2. *If $Y \subseteq X$ then* $\dim(X) = \dim(Y) + \operatorname{codim}_X(Y)$.
3. *If f is a non-zero, non-unit element of $k[X]$, then* $\dim V_X(f) = \dim(X) - 1$.
4. *If $f_1, \ldots, f_r$ are elements of $k[X]$, then* $\dim V_X(f_1, \ldots, f_r) \geq \dim(X) - r$.

Proof. (1) This can be shown using Noether normalization.

(2) This follows because k-algebra domains satisfy the catenary property: all maximal chaine of primes have the same length. It follows that

$$\begin{aligned} \dim k[X] &= \dim \frac{k[X]}{I(Y)} + \operatorname{ht} I(Y) \\ \dim k[X] &= \dim k[Y] + \operatorname{ht} I(Y) \\ \dim(X) &= \dim(Y) + \operatorname{codim}_X(Y). \end{aligned}$$

(3) By Krull's Principal Ideal Theorem $\operatorname{ht}((f)) \leq 1$. Since $(0) \subsetneq (f) \subsetneq k[X]$ in fact we have $\operatorname{ht}((f)) = 1$, so by Proposition 3.25 every irreducible component of $V_X(f)$ has codimension 1 in X. Equivalently, by (2), $\dim V_X(f) = \dim(X) - 1$.

(4) By Krull's Height Theorem $\operatorname{ht}((f_1, \ldots, f_r)) \leq r$ so by Proposition 3.25 some irreducible component of $V_X((f_1, \ldots, f_r))$ has codimension $\leq r$ and therefore, by (2), it has dimension $\geq \dim(X) - r$. Thus $\dim V_X(f) \geq \dim(X) - r$. □

Remark 3.9. If $f_1, \ldots, f_r$ are elements of $k[x_1, \ldots, x_n]$ such that $\dim V^{\mathbb{A}}(f_1, \ldots, f_r) = n - r$ we say that $V^{\mathbb{A}}(f_1, \ldots, f_r)$ is a *complete intersection*.

We now turn to the dimension of quasi-affine varieties. We see below that if U is quasi-affine, then $\dim(U) = \dim(\overline{U})$.

Proposition 3.10. *If U is a dense open set of an affine variety X, then* $\dim(U) = \dim(X)$. *In particular,* $\dim(U) = \dim(\overline{U})$.

Proof. Suppose that $Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n$ is a sequence of distinct closed irreducible subsets of U . Let Z_i be the Zariski closure of Y_i in X for $0 \leq i \leq n$. As Y_i is closed in U, we have $Y_i = U \cap W$ for some closed W in X, which means that $Y_i \subseteq W$ and thus $Z_i \subseteq W$ as Z_i is the smallest closed set of X that contains Y_i. Then the containments

$$Y_i = U \cap Z_i \subseteq U \cap W = Y_i$$

ensure $Z_i \cap U = Y_i$. Thus

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \tag{3.1}$$

is a sequence of distinct closed irreducible subsets of X, yielding $\dim(U) \leq \dim(X)$.

In particular, $\dim(U)$ is finite, so we can choose a maximal chain of Y_i's. Since the chain is maximal, Y_0 is a point and $Y_n = U$. Now if W is an irreducible closed subset of X such that the open subset $W \cap U$ of W is nonempty, we then have that $W \cap U$ is dense in W. Indeed, $W = \overline{W \cap U} \cup (W \cap (X \setminus U))$ is a union of closed sets in W and the latter is not equal to W provided $W \cap U \neq \emptyset$, so it must be that $W = \overline{W \cap U}$.

In particular, if $A \subsetneq B$ are irreducible closed subsets of X such that $A \cap U \neq \emptyset$ and $A \cap U = B \cap U$, then we have that $A = \overline{A \cap U} = \overline{B \cap U} = B$. Thus we have that (3.1) is a maximal chain in X, and hence by the catenary property $\dim(Y) = \dim(X)$. □

Recall from Math 905 that if R is a k-algebra domain then $\dim(R) = \operatorname{trdeg}_k(\operatorname{Frac}(R))$. Building upon this we show that dimension is a birational invariant.

Theorem 3.11. *Let X and Y be irreducible affine varieties that are birational to each other. Then* $\dim(X) = \dim(Y)$.

Proof. By Corollary 1.98 a birational map $X \to Y$ induces an isomorphism of fraction fields $\operatorname{Frac}(k[X]) = k(X) \cong k(Y) = \operatorname{Frac}(k[Y])$. Thus

$$\dim(X) = \dim k[X] = \operatorname{trdeg}_k(\operatorname{Frac}(k[X])) = \operatorname{trdeg}_k(\operatorname{Frac}(k[Y])) = \dim k[Y] = \dim(Y).$$

□

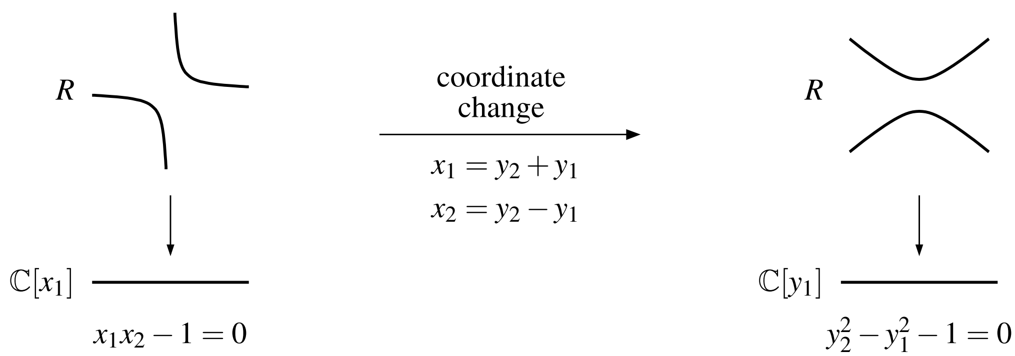
Wednesday, March 3

3.2 Finite maps

We start by recalling Noether normalization.

Example 3.12. (Idea behind Noether normalization) Let $R = k[x_1, x_2]/(x_1x_2 - 1)$ be the coordinate ring of the variety $X = V(x_1x_2 - 1) \subset \mathbb{A}^2_k$. Then R is not integral (and hence not module-finite) over $A = k[x_1]$. We can see this from the figure below on the left: the map induced by $A \hookrightarrow R$ is the projection onto the x_1-axis. It can be seen geometrically that this map does not satisfy the Lying Over property for the origin.

It is easy to change the situation however by a linear coordinate transformation: if we set e. g. $x_1 = y_2 + y_1$ and $x_2 = y_2 - y_1$ then we can write R also as $R' = k[y_1, y_2]/(y_2^2 - y_1^2 - 1)$, and now R' is integral, hence module-finite, over $A' = k[y_1]$ since the polynomial $y_2^2 - y_1^2 - 1$ is monic in y_2. Geometrically, the coordinate transformation has tilted the algebraic set X as in the picture above on the right so that e. g. the Lying Over property now obviously holds. Note that this is not special to the particular transformation that we have chosen; in fact, almost any linear coordinate change would have worked to achieve this goal.



Recall from Math 905:

Theorem 3.13 (Noether Normalization)**.** *Let k be a field (not necessarily algebraically closed), and R be a finitely generated k-algebra. Then, there are $x_1, \ldots, x_d \in R$ algebraically independent over K such that $k[x_1, \ldots, x_d] \hookrightarrow R$ is module-finite.*

(Projective Noether Normalization) *Let k be an infinite field (this holds if k is algebraically closed), and R be a finitely generated standard-graded k-algebra with $R_0 = k$. Then there are homogeneous* linear *elements $x_1, \ldots, x_d \in R_1$ algebraically independent over k such that R is module-finite over $k[x_1, \ldots, x_d]$.*

The ring $k[x_1, \ldots, x_d]$ in Theorem 3.13 is called a *Noether normalization* for R.

We further have from Math 905

Theorem 3.14. *If R is a finitely generated k-algebra, then* $\dim(R) = d$ *such that there exists a Noether Normalization $k[x_1, \ldots, x_d]$ of R.*

We now interpret the algebraic notion of a ring extension being finite geometrically, in particular building a geometric framework for Noether normalization.

Definition 3.15. Suppose that $f : X \to Y$ is a regular map of affine varieties. We say that f is a *finite map* if $f^* : k[Y] \to k[X]$ is integral, that is, $k[X]$ integral and thus a finitely generated module over the subring $f^*(k[Y])$.

In the case when f is dominant and thus f^* is injective, it may sometimes be convenient to abuse notation and identify $k[Y]$ with its isomorphic image $f^*(k[Y])$ viewing $k[Y]$ as a subring of $k[X]$.

Example 3.16. The projection of the parabola $V^{\mathbb{A}}(y_2^2 - y_1^2 - 1)$ onto the axis $V^{\mathbb{A}}(y_2)$ in Example 3.12 is a finite map as it induces and integral extension of coordinate rings.

Theorem 3.17 (Properties of finite maps of affine varieties)**.** *Suppose that $f : X \to Y$ is a finite map of affine varieties. Then*

1. *$f^{-1}(\mathbf{b})$ is a finite set for all $\mathbf{b} \in Y$,*
2. *if f is dominant then f is surjective,*
3. *f is closed.*

Proof. (1) Let $k[X] = k[x_1, \ldots, x_n]/I(X)$. It suffices to show that each x_i assumes only finitely many values on $f^{-1}(\mathbf{b})$. Since $k[X]$ is integral over $f^*(k[Y])$ each x_i satisfies an integral dependence relation

$$x_i^m + f^*(y_{m-1})x_i^{m-1} + \cdots + f^*(y_0) = 0$$

for some $y_0, \ldots, y_{m-1} \in k[Y]$. Suppose $\mathbf{a} \in f^{-1}(\mathbf{b})$ and note this means $f(\mathbf{a}) = \mathbf{b}$ and hence $f^*(y_i)(\mathbf{a}) = y_i(f(\mathbf{a})) = y_i(\mathbf{b})$. Plugging $\mathbf{a}$ into the displayed equation gives

$$a_i^m + b_{m-1}a_i^{m-1} + \cdots + b_0 = 0.$$

Thus a_i must be one of the $\leq m$ roots of the above equation. We have obtained that each coordinate of $\mathbf{a}$ has finitely many possible values, thus $f^{-1}(\mathbf{b})$ is a finite set.

The proof of (2) and (3) is omitted. □

Theorem 3.18. *Suppose that X is an affine variety. Then there exists a dominant finite map $f : X \to \mathbb{A}_k^d$ for a unique d, namely $d = \dim(X)$.*

Proof. There exist, by Theorem 3.13, $x_1, \ldots, x_d \in k[X]$ algebraically independent over k such that $k[x_1, \ldots, x_d] \hookrightarrow k[X]$ is module-finite and hence $d = \dim k[X] = \dim(X)$. Define a regular map $f : X \to \mathbb{A}_k^d$ by $f(\mathbf{a}) = (x_1(\mathbf{a}), x_2(\mathbf{a}), \ldots, x_r(\mathbf{a}))$ for $\mathbf{a} \in X$. Let $t_1, \ldots, t_d$ be the coordinate functions on $\mathbb{A}_k^d$. Then $f^* : k[\mathbb{A}_k^d] \to k[X]$ is the k-algebra homomorphism defined by $f^*(t_i) = x_i$ for $1 \leq i \leq r$. Thus f^* is injective and $k[X]$ is integral over $k[\mathbb{A}^d]$, and so f is dominant by Lemma 1.95 and finite.

Given a dominant finite map $f : X \to \mathbb{A}_k^d$, its pullback $f^* : k[\mathbb{A}_k^d] \to k[X]$ is an integral injection by Lemma 1.95 and thus $d = \dim k[\mathbb{A}_k^d] = \dim k[X] = \dim(X)$. □

Example 3.19. As the identity map $\mathbb{A}_k^d \to \mathbb{A}_k^d$ is a finite map, we conclude that $\dim \mathbb{A}_k^d = d$.

Exercise 3.20. Show that if $V \subseteq \mathbb{A}^n_k$ is a k-vector subspace of k^n, hence also an affine variety then $\dim V = \dim_k V$, that is, the dimension of V as a an affine variety coincides with its dimension as a vector space.

Example 3.21. The converse of Theorem 3.17 (1) is false: the inclusion $\iota : \mathbb{A}^1_k \setminus \{0\} \to \mathbb{A}^1_k$ has finite fibers, but the pullback $\iota^* : k[\mathbb{A}^1_k] = k[x] \to k[A^1_k \setminus \{0\}] = k[x, x^{-1}]$ is not module-finite.

Friday, March 7

3.3 Dimension of projective varieties

The topological Definition 3.1 of dimension and codimension is in effect for projective varieties. We now want to find equivalent algebraic definitions.

Definition 3.22. Let X be an irreducible projective variety. The *field of rational functions* on X is the set

$$k(X) = \left\{ \frac{F}{G} \mid F, G \in k[X] \text{ homogeneous of the same degree }, G \neq 0 \right\}.$$

Exercise 3.23.

1. Verify that $k(X)$ is a proper subfield of the fraction field of $k[X]$.
2. Verify that $k(X)$ is the set of degree 0 elements in $S^{-1}k[X]$, where S is the set of nonzero homogeneous elements in $k[X]$. We consider the degree of an element $a/b \in S^{-1}k[X]$ to be $\deg(a) - \deg(b)$.

Lemma 3.24. *Let X be an irreducible projective variety, fix $0 \neq t \in k[X]_1$ and let S be the set of nonzero homogeneous elements in $k[X]$. Then there is an isomorphism of graded rings*

$$S^{-1}k[X] \cong k(X)[t, t^{-1}] = \bigoplus_{i \in \mathbb{Z}} k(X)t^i.$$

Proof. A homogeneous element of $S^{-1}k[X]$ degree d has an expression a/b, where $a \in k[X]_i, b \in k[X]_j$ and $i - j = d$. As in Exercise 3.23 we see that the set of elements of degree zero in $S^{-1}k[X]$ is $k(X)$. Moreover, we can rewrite

$$\frac{a}{b} = \frac{a}{t^i} \cdot \frac{t^j}{b} \cdot t^{i-j}$$

where the first two factors are in $k(X)$, hence their product is too. This shows that $[S^{-1}k[X]]_d = k(X)t^d$ is the degree d part of $k(X)[t, t^{-1}]$, whence the claimed isomorphism follows. $\square$

Proposition 3.25. *The dimension of an projective variety X is equal to all of the following numbers (in particular* $\dim(X)$ *is finite)*

1. *the dimension of any dense open subset U of X*

2. *the transcendence degree of the faction field $k(X)$ of the coordinate ring* $\operatorname{trdeg}_k k(X)$, *provided X is irreducible*

3. $\dim k[X] - 1$

The codimension of a closed subset Y of X is the height of the corresponding radical ideal $I(Y)$.

Proof. 1. The proof of (1) is identical to Proposition 3.10.
2. The proof will be added later.
3. By Lemma 3.24 $S^{-1}k[X] \cong k(X)[t, t^{-1}] = \bigoplus_{i\in\mathbb{Z}} k(X)t^i$ as graded rings. Thus

$$\operatorname{Frac}(k[X]) = \operatorname{Frac}(S^{-1}k[X]) = \operatorname{Frac}(k(X)[t, t^{-1}]) = k(X)(t),$$

where $k(X)(t)$ is the smallest sub-field of $\operatorname{Frac}(k[X])$ that contains both $k(X)$ and t. Note that t is transcendental over $k(X)$ since a degree 1 element cannot be a root of a polynomial with degree zero coefficients. We conclude from (2) that

$$\dim k[X] = \operatorname{trdeg}_k(\operatorname{Frac}(k[X])) = \operatorname{trdeg}_k(k(X)(t)) = \operatorname{trdeg}_k k(X) + 1 = \dim(X) + 1.$$

□

Example 3.26. The dimension of $\mathbb{P}^n_k$ is n since it has dense open sets $U_i \cong \mathbb{A}^n_k$.

Example 3.27. The dimension of Grassmannian varieties is given by $\dim \operatorname{Gr}(d, n) = d(n-d)$ by Proposition 3.25 part (1) and Theorem 2.72 part (3).

Example 3.28. Since the dimension is an isomorphism invariant, the dimension of the Veronese variety is $\dim V_{n,d} = \dim \mathbb{P}^n_k = n$ and that of the Segre variety is $\dim \Sigma_{m,n} = \dim \mathbb{P}^m_k \times \mathbb{P}^n_k = m + n$.

3.4 The theorem on dimension of fibers

In the previous section we have discussed the dimension of projective varieties. This informs us what the dimension of quasi-projective varieties is as well, since if U is an open set of some projective variety, then $\dim(U) = \dim \overline{U}$ where the latter is a projective variety.

Definition 3.29. A real-valued function $e : X \to \mathbb{R}$ from a topological space X is *upper-semicontinuous* if for each $m \in \mathbb{R}$, the following subset is open in X:

$$U_m = \{x \in X \mid e(x) < m\}.$$

Equivalently, the following set is closed

$$V_m = \{x \in X \mid e(x) \geq m\}.$$

Intuitively this means that the value of e can only increase “in the limit”.

Example 3.30. The function $e : \mathbb{R} \to \mathbb{R}$, $e(x) = \lfloor x \rfloor$ is upper-semicontinuous. Indeed for $m \in \mathbb{R}$ we have

$$U_m = \{x \in X \mid e(x) < m\} = (-\infty, \lceil m \rceil) \text{ is open.}$$

We now state the theorem on dimension of fibers.

Theorem 3.31. *Let X, Y be quasi-projective varieties and $f : X \to Y$ a dominant regular map. For $\mathbf{a} \in X$ set $F_{\mathbf{a}} := f^{-1}(f(\mathbf{a}))$, which we call the* fiber *of $f(\mathbf{a})$, and $\mu(\mathbf{a}) = \dim_{\mathbf{a}} F_{\mathbf{a}}$ which we define to be the maximum dimension of an irreducible component of $F_{\mathbf{a}}$ containing $\mathbf{a}$. For $\mathbf{b} \in Y$ we set $\lambda(\mathbf{b}) = \dim f^{-1}(\mathbf{b})$. Then*

1. *the function $\mu : X \to \mathbb{R}$ is upper-semicontinuous with respect to the Zariski topology, i.e., for each $m \in \mathbb{N}$ the set*

$$V_m = \{\mathbf{a} \mid \mu(\mathbf{a}) \geq m\} \textit{ is closed in } X.$$

2. *if X is irreducible and μ_{min} is the minimum value attained by μ then the set*

$$U = \{\mathbf{a} \in X \mid \mu(\mathbf{a}) = \mu_{min}\} \textit{ is a dense open set of } X$$

   and $\mu_{min} = \dim(X) - \dim(Y)$.

3. *if X, Y are projective varieties and f is surjective, then the function $\lambda : Y \to \mathbb{R}$ is upper-semicontinuous with respect to the Zariski topology, i.e., for each $m \in \mathbb{N}$ the set*

$$Z_m = \{\mathbf{b} \mid \lambda(\mathbf{b}) \geq m\} \textit{ is closed in } Y.$$

4. *if X, Y are projective varieties, f is surjective, X is irreducible and λ_{min} is the minimum value attained by λ then $\lambda_{min} = \dim(X) - \dim(Y)$.*

Remark 3.32. We say that a *general* element (point) in an irreducible variety X satisfies some property P if $P(\mathbf{a})$ is true for all $\mathbf{a}$ in some nonempty, hence dense, open set of X.

Thus Theorem 3.31 part (2) can be stated as the dimension of the fiber at a general point is $\dim X - \dim Y$.

Example 3.33. Consider the Segre surface $Q = V^{\mathbb{P}}(XZ - YW) \subseteq \mathbb{P}^3$ which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$; see Example 2.11. Project from a point $\mathbf{b}$ of Q onto a plane. Then we see that

$$F_{\mathbf{a}} = \begin{cases} \{\mathbf{a}\} & \text{if } \mathbf{a}, \mathbf{b} \text{ are not on the same line in a ruling on } Q \\ L & \text{if } \mathbf{a}, \mathbf{b} \text{ are on the same line } L \text{ in a ruling on } Q. \end{cases}$$

Thus $\mu_{min} = 0$ and $U = Q \setminus \{L_1, L_2\}$, where L_1, L_2 are the two lines on Q through $\mathbf{b}$.

Belo is an important corollary of the theorem which uses the following exercise.

Exercise 3.34. If $Y \subseteq X$ are both affine or both projective varieties then $\dim(Y) \leq \dim(X)$ and $\dim(Y) = \dim(X)$ if and only if $Y = X$.

Corollary 3.35. *Let X, Y be projective varieties with Y irreducible. Suppose for every $\mathbf{b} \in Y$ the fiber $f^{-1}(\mathbf{b})$ is irreducible and all the fibers have the same dimension d. Then X is irreducible.*

Proof. Let $X = \bigcup_{i=1}^n X_i$ be the irreducible decomposition of X and consider restrictions $f_i = F|_{X_i}$ and functions $\lambda_i : Y \to \mathbb{R}, \lambda_i(\mathbf{b}) = \dim f_i^{-1}(\mathbf{b})$. Then since $f^{-1}(\mathbf{b}) = \bigcup_{i=1}^n f_i^{-1}(\mathbf{b})$ with $f^{-1}(\mathbf{b})$ irreducible and $f_i^{-1}(\mathbf{b})$ closed, it follows that for every $\mathbf{b} \in Y$ $f^{-1}(\mathbf{b}) = f_i^{-1}(\mathbf{b})$ for some i (here i depends on $\mathbf{b}$). We aim to show that the same i works for every $\mathbf{b}$.

Since every function λ_i is upper semi-continuous by Theorem 3.31, we have that for each i the set $Y_i := \{\mathbf{b} \in Y \mid \lambda_i(\mathbf{b}) = d\}$ is closed in Y and by the previous considerations $Y = \bigcup_{i=1}^n Y_i$. Now Y is irreducible, so $Y = Y_j$ for some j. By the definition of Y_j this implies that all fibers of f_j have dimension d and thus, by the Exercise 3.34, since $f_j^{-1}(\mathbf{b}) \subseteq f^{-1}(\mathbf{b})$ and $f^{-1}(\mathbf{b})$ is irreducible it follows that $f_j^{-1}(\mathbf{b}) = f^{-1}(\mathbf{b})$ for every $\mathbf{b} \in Y$.

Finally this shows that $X = f^{-1}(Y) = f_j^{-1}(Y) = X_j$ and hence that X is irreducible. □

The theorem on dimension of fibers is often applied in conjunction with incidence correspondences. Vaguely, given varieties X, Y an incidence correspondence is a set of the form

$$\Sigma = \{(x, y) \mid x \in X, y \in Y, x \text{ is related to } y \text{ in a specific manner}\} \subseteq X \times Y.$$

Example 3.36 (The universal d-plane)**.** Consider

$$\Sigma = \{(\mathbf{a}, V) \mid [\mathbf{a}] \in \mathbb{P}_k^n, V \in \mathrm{Gr}(d+1, n+1), \mathbf{a} \in V\} \subseteq \mathbb{P}_k^n \times \mathrm{Gr}(d+1, n+1).$$

Then Σ is a variety since the condition that $\mathbf{a} \in V$ is polynomial in the coordinates of $\mathbf{a}$ and the Plücker coordinates of V. Namely, $\mathbf{a} \in V$ if and only if $\mathrm{Span}_k\langle \mathbf{a} \rangle + V$ is a vector space of dimension $d+1$ oif and only if the $(d+2) \times (d+2)$ minors of the matrix obtained by stacking any matrix A whose rows are a basis of V and the row vector $\mathbf{a}$ are equal to 0. But by Laplace expansion along the row $\mathbf{a}$, these minors are linear combinations of the entries of $\mathbf{a}$ with coefficients given by the $(d+1) \times (d+1)$ minors of A, i.e., the Plucker coordinates of V.

We will prove below that for all $n \geq 0$ the Grassmannian varieties $\mathrm{Gr}(d, n)$ are irreducible by induction on d. The cases $d = 0$ is clear since the only 0-dimensional vector space is the zero vector space. Thus $\mathrm{Gr}(d, n)$ is a point, which is irreducible. Now suppose that for all $n \geq 0$ we know that $\mathrm{Gr}(d, n)$ is irreducible. We will show that $\mathrm{Gr}(d+1, n+1)$ is irreducible.

To compute the dimension of Σ we can proceed in two ways:

- the fiber of the projection $\pi_1 : \Sigma \to \mathbb{P}^n_k$ onto the first factor over any point $[\mathbf{a}] \in \mathbb{P}^n_k$ is given by the set of d-dimensional subspaces of $k^{n+1}/\operatorname{Span}_k\langle \mathbf{a}\rangle$ and this is isomorphic to $\operatorname{Gr}(d,n)$. By Corollary 3.35 and the inductive hypothesis, since all fibers are the same and they are irreducible we conclude that Σ is irreducible. By part (2) of Theorem 3.31 we deduce

$$\dim \Sigma = \dim \mathbb{P}^n + \dim \operatorname{Gr}(d,n) = n + d(n-d).$$

- the fiber of the projection $\pi_2 : \Sigma \to \operatorname{Gr}(d+1, n+1)$ onto the second factor over any point $V \in \operatorname{Gr}(d, n+1)$ is given by V, which is a $d+1$-dimensional vector space, but d dimensional as a projective variety; specifically $V \cong \mathbb{P}^d_k$. By Theorem 3.31 (2) we have

$$\dim \Sigma = \dim \operatorname{Gr}(d+1, n+1) + \dim \mathbb{P}^d_k = (d+1)(n-d) + d.$$

  Notice that π_2 is surjective and continuous. Since $\operatorname{Gr}(d+1, n+1)$ is the image of an irreducible variety Σ through a continuous map, $\operatorname{Gr}(d+1, n+1)$ is irreducible.

We have thus proven:

Corollary 3.37. *Grassmannians are irreducible projective varieties.*

Wednesday, March 12

We now start the proof of Theorem 3.31. We will need the following lemma.

Lemma 3.38 (Points are set-theoretic complete intersections)**.** *Suppose that Y is an quasi-projective variety of dimension $d \geq 1$ and $\mathbf{b} \in Y$. Then there exist an affine neighborhood W in Y and $f_1, \dots, f_d \in \mathcal{O}_Y(W)$ such that $V^{\mathbb{A}}_W(f_1, \dots, f_d) = \{\mathbf{b}\}$.*

Proof. Recall that $\mathcal{O}_Y(\mathbf{b}) = k[W]_{I(\mathbf{b})}$ is a local ring with maximal ideal $I(\mathbf{b})\mathcal{O}_Y(\mathbf{b})$. From Math 905 we know that in a local ring there exists a system of parameters $f_1, \dots, f_s$ such that $s = \dim \mathcal{O}_Y(\mathbf{b}) = \dim k[W] \leq \dim Y$ and $\sqrt{(f_1, \dots, f_s)} = I(\mathbf{b})\mathcal{O}_Y(\mathbf{b})$. We may assume $s = d$ by adding redundant generators and we may assume $f_1, \dots, f_s \in \mathcal{O}_Y(W)$ by clearing denominators. Now the claim follows since $\sqrt{(f_1, \dots, f_s)} = I(\mathbf{b})$ in $\mathcal{O}_Y(W)$. □

We next prove the following version of item (4) in Theorem 3.31.

Theorem 3.39. *Let $f : X \to Y$ be a dominant regular map between irreducible quasi-projective varieties. Then $\dim Y \leq \dim X$ and:*

1. *Suppose that $\mathbf{b} \in \operatorname{Im}(f)$. Then $\dim f^{-1}(\mathbf{b}) \geq \dim X - \dim Y$.*
2. *There exists a nonempty open subset U of Y such that $\dim f^{-1}(\mathbf{b}) = \dim X - \dim Y$ for all $\mathbf{b} \in U$.*

Proof. 1. Applying Lemma 3.38 we can find an affine open subset $W \subseteq Y$ and $g_1, \ldots, g_d \in k[W]$ so that $V_W^{\mathbb{A}}(g_1, \ldots, g_d) = \{\mathbf{b}\}$ where $d = \dim Y$. Set $f^{-1}(W) = U$, which is a nonempty open set of X. Because X is irreducible U is dense which gives $\dim X = \dim U$. Now we claim

$$f^{-1}(\mathbf{b}) = V_U(f^*(g_1), \ldots, f^*(g_d)).$$

Indeed,

$$\begin{aligned} \mathbf{a} \in f^{-1}(\mathbf{b}) \quad &\iff \quad f(\mathbf{a}) = \mathbf{b} \iff g_i(f(\mathbf{a}) = 0, \forall 1 \le i \le d \\ &\iff \quad \mathbf{a} \in V_U(f^*(g_1), \ldots, f^*(g_d)). \end{aligned}$$

Krull's height theorem shows $\operatorname{codim}_U f^{-1}(\mathbf{b}) = \operatorname{ht}(f^*(g_1), \ldots, f^*(g_d)) \le d = \dim Y$. Thus

$$\dim f^{-1}(\mathbf{b}) = \dim U - \operatorname{codim}_U f^{-1}(\mathbf{b}) \ge \dim X - \dim Y.$$

2. We may again replace Y with an affine open subset W, X by an affine open subset $V \subset f^{-1}(W)$, and f by $f|_V : V \to W$. Since f is dominant, f determines an inclusion $f^* : k[W] \to k[V]$, hence an inclusion $k(W) = k(Y) \xrightarrow{f^*} k(X) = k(V)$. Consider the subring R of $k(V)$ generated by $k(W)$ and $k[V]$. This is a domain which is a finitely generated $k(W)$-algebra.

By Noether's normalization Theorem 3.13 we have that there exist $t_1, \ldots, t_r$ in R such that $t_1, \ldots, t_r$ are algebraically independent over $k(W)$ and R is integral over the polynomial ring $k(W)[t_1, \ldots, t_r]$. We may assume, after multiplying by an element of $k[W]$, that $t_1, \ldots, t_r \in k[V]$. Since the fraction field of R is $k(V)$ we have

$$r = \operatorname{trdeg}_{k(W)} k(V) = \operatorname{trdeg}_k k(V) - \operatorname{trdeg}_k k(W) = \dim X - \dim Y.$$

Now consider in $k[V]$ the subring $f^*(k[W])[t_1, \ldots, t_r] = k[W] \otimes_k k[t_1, \ldots, t_r] = k[W \times \mathbb{A}_k^r]$. We have factorization of f of the form

$$V \xrightarrow{\varphi} W \times A_k^r \xrightarrow{\pi} W, \qquad \mathbf{a} \mapsto (f(\mathbf{a}), t_1(\mathbf{a}), \ldots, t_r(\mathbf{a})) \mapsto f(\mathbf{a}).$$

Note that $\pi^{-1}(\mathbf{b}) = \{\mathbf{b}\} \times \mathbb{A}_k^r \cong \mathbb{A}_k^r$ has dimension r and by the closed mapping theorem $\varphi(f^{-1}(\mathbf{b}))$ is a subvariety of $\pi^{-1}(\mathbf{b})$, thus $\dim \varphi(f^{-1}(\mathbf{b})) \le r$. We'll show that φ is a finite map when further restricted to an open set. This will yield that $\dim f^{-1}(\mathbf{b}) = \dim \varphi(f^{-1}(\mathbf{b})) \le r = \dim X - \dim Y$, finishing the proof.

Now to show that a restriction of φ is finite we proceed as follows: We have that $k[V]$ is a finitely generated k-algebra, so it is a finitely generated $f^*(k[W])[t_1, \ldots, t_r]$-algebra, say generated by $v_1, \ldots, v_l$ as a $f^*(k[W])[t_1, \ldots, t_r]$-algebra. Since R is integral over $f^*(k(W))[t_1, \ldots, t_r]$ there exist polynomials $F_i(x)$ in the indeterminate x, such that $F_i(v_i) = 0$ for $1 \le i \le l$. The coefficients of the F_i belong to $f^*(k(W))[t_1, \ldots, t_r]$, so they have denominators in $f^*(k[W])$. let $g \in f^*(k[W])$ be a common denominator of all the coefficients in all the F_i's and let $h \in k[W]$ be such that $f^*(h) = g$.Then $k[V][1/g]$ is integral over $f^*(k[W, 1/h])[t_1, \ldots, t_r]$.

Let $U = D_W(h)$. The considerations above show that the map $\varphi : f^{-1}(U) \to U \times \mathbb{A}^r_k$ is finite, as desired. Then as described above it follows that for $\mathbf{b} \in U$ we have $\dim f^{-1}(\mathbf{b}) \leq r = \dim X - \dim Y$. In part (1) we have shown the opposite inequality $\dim f^{-1}(\mathbf{b}) \geq \dim X - \dim Y$. Thus $\dim f^{-1}(\mathbf{b}) = \dim X - \dim Y$ for $\mathbf{b} \in U$. □

We return to the proof of Theorem 3.31 and prove a version of part (3).

Theorem 3.40. *Let $f : X \to Y$ be a dominant regular map between irreducible quasi-projective varieties. Then for every $m \in \mathbb{R}$ the following set is Zariski closed*

$$Z_m = \{\mathbf{b} \in \mathrm{Im}(f) : \dim f^{-1}(\mathbf{b}) \geq m\}.$$

Thus the map $\lambda : \mathrm{Im}(f) \to \mathbb{R}, \lambda(\mathbf{b}) = \dim f^{-1}(\mathbf{b})$ is upper-semicontinuous.

Proof. It suffices to prove the claim for $m \in \mathbb{Z}$ as $Z_m = Z_{\lceil m \rceil}$.

Let λ_{min} be the smallest value attained by λ. In Theorem 3.39 we showed $\lambda_{min} = \dim X - \dim Y$ and also that there is a nonempty open set $U \subseteq Y$ such that $\lambda(\mathbf{b}) = \lambda_{min}$ for $\mathbf{b} \in U$. We have however *not* shown that U is the set of *all* points where λ attains its minimum value.

Claim 3.41. The set $U_{min} = \{\mathbf{b} \in Y : \dim f^{-1}(\mathbf{b}) = \lambda_{min}\}$ is Zariski open.

Set $U_0 = U$ and $V_0 = Y \setminus U_0$. If λ takes values strictly greater than λ_{min} on V_0 then we have $U_{min} = U_0$ is open. Else applying Theorem 3.39 to the map $f|_{f^{-1}(V_0)}$ which has the same λ_{min} we can find an open set $U_1 \subseteq V_0$ on which $\lambda(\mathbf{b}) = \lambda_{min}, \forall \mathbf{b} \in U_1$. This process has to terminate else there is an infinite ascending chain of open sets

$$U_0 \subsetneq U_0 \cup U_1 \subsetneq U_0 \cup U_1 \cup U_2 \cdots$$

which contradicts the Noetherianity of Y. Assuming this process terminates after n steps we observe that $U_{min} = U_0 \cup U_1 \cup \cdots \cup U_n$ is open.

Furthermore the set $Y \setminus U = V_{\lambda_{min}+1}$ is closed. Now we can restrict f to $f^{-1}(V_{\mu_{min}+1})$ where λ has a new minimum value $\lambda' \geq \lambda_{min} + 1$ and repeat the argument above to find an open set where the value λ' is attained and a closed set $V_{\lambda'+1}$ where it is exceeded. It follows that $V_m = V_{\lambda_{min}+1}$ is closed for all $\lambda_{min} + 1 \leq m \leq \lambda'$ and properly contains $V_{\lambda'+1}$. By Noetherianity of Y eventually $V_m = \emptyset$ for large enough m. □

Friday, March 14 (π Day)

Today we look at an extended example of applying the Theorem on Dimension of Fibers. The goal is to decide roughly how many lines there are on a surface in three-dimensional space, meaning are there no lines, finitely many lines or infinitely many lines.

A surface S in $\mathbb{P}^3_k$ is the set of zeros of a homogeneous polynomial $F(X_0, X_1, X_2, X_3)$, denoted $S = V^{\mathbb{P}}(F)$ and a line L in $\mathbb{P}^3_k$ consists of the equivalence classes with respect to $\sim$ of nonzero points in a 2-dimensional vector space V in k^4 .

Theorem 3.42. *If $S = V^{\mathbb{P}}(F)$ is a surface in $\mathbb{P}^3_k$ and $\mathrm{char}(k) \neq 2$ then*

$$S \text{ contains } \begin{cases} \textit{inifinitely many lines} & \textit{if } \deg(F) \leq 2 \\ \textit{finitely many lines} & \textit{if } \deg(F) = 3 \textit{ and } F \textit{ is general} \\ \textit{no lines} & \textit{if } \deg(F) \geq 4 \textit{ and } F \textit{ is general.} \end{cases}$$

Proof. If F is linear $S \cong \mathbb{P}^2_k$ is a plane, thus contains infinitely many lines.

If F is quadratic, then S is projectively equivalent to a variety defined by a quadratic polynomial in *standard form*

$$F(X_0, X_1, X_2, X_3) = c_0X_0^2 + c_1X_1^2 + c_2X_2^2 + c_3X_3^2 \text{ with each } c_i \in \{0, 1\}.$$

There are several possibilities:

- if three of the c_i's are zero then S is a plane and thus contains infinitely many lines.
- if exactly two of the c_i's are zero then $S \cong V^{\mathbb{P}}(X_0^2 + X_1^2) = V^{\mathbb{P}}((X_0 + iX_1)(X_0 - iX_1))$ is the union of two planes and thus contains infinitely many lines.
- if only one c_i is zero then S is a double cone and thus contains infinitely many lines.
- if all $c_i = 1$ then we can write

  $$F(X_0, X_1, X_2, X_3) = (X_0 + iX_1)(X_0 - iX_1) + (X_2 + iX_3)(X_2 - iX_3) = XZ - YW$$

  This yields that S is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$, which contains two (infinite) families of lines.

Now suppose that $F(X_0, X_1, X_2, X_3)$ is a cubic (degree three) homogeneous polynomial. Consider the incidence correspondence

$$\Sigma = \{(L, F) \mid L \text{ line}, F \text{ homogeneous of degree } 3, L \subseteq V^{\mathbb{P}}(F)\} \subseteq \mathrm{Gr}(2, 4) \times \mathbb{P}^{19},$$

where the set of lines $L \subseteq \mathbb{P}^3_k$ is $\mathrm{Gr}(2, 4)$ and $F = \sum_{i_0+i_1+i_2+i_3=3} c_{i_0i_1i_2i_3} X_0^{i_0} X_1^{i_1} X_2^{i_2} X_3^{i_3}$ is identified with the point $[\cdots : c_{i_0i_1i_2i_3} : \cdots] \in \mathbb{P}^{19}$ (there are 20 monomials of degree 3 in 4 variables, so 20 coefficients in F which give a point in $\mathbb{P}^{19}$).

We start by analyzing the projection $\pi_1 : \Sigma \to \mathrm{Gr}(2, 4)$ to the first factor. All lines are projectively equivalent, so up to projective equivalence $L = \mathbb{V}^{\mathbb{P}}(X_0, X_1)$ and also all fibers of π_1 are projectively equivalent. The fiber

$$f^{-1}(L) = \{F \mid \mathbb{V}^{\mathbb{P}}(X_0, X_1) \subseteq \mathbb{V}^{\mathbb{P}}(F)\} = \{F \mid F \in (X_0, X_1)\}.$$

The condition $F \in (X_0, X_1)$ means that the coefficients of $X_2^3, X_2^2X_3, x_2X_3^2, X_3^3$ in F are all zeros and the other 16 coefficients are arbitrary (not all 0). So $f^{-1}(L) \cong \mathbb{P}^{15}$ has dimension 15.

Since every fiber is non-empty, the projection π_1 is surjective. Since all fibers are irreducible of the same dimension, Σ is irreducible by Corollaries 3.35 and 3.37 and by the Theorem on Dimension of Fibers we compute

$$\dim(\Sigma) = \dim \mathrm{Gr}(2, 4) + \dim f^{-1}(L) = 4 + 15 = 19.$$

We continue by analyzing the projection $\pi_2 : \Sigma \to \mathbb{P}^{19}$ to the second factor. Note that we *don't* know whether π_2 is surjective at this point (we don't know if a cubic surface must contain a line) but we can set $Z = \mathrm{Im}(\pi_2)$, which is a projective variety by the Closed Mapping Thorem and we can look at $\pi_2 : \Sigma \to Z$.

Consider the line $L_0 = \mathbb{V}^{\mathbb{P}}(X_2, X_3)$. It is contained in the open set U of $\mathrm{Gr}(2,4)$ given by the following matrices or corresponding lines

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix} \leftrightarrow L = \{[s : t : as + ct : bs + dt] \mid, s, t \in k\}.$$

Here L_0 corresponds to $a = b = c = d = 0$. Observe that L_0 is the only line in this neighborhood that lies on $V^{\mathbb{P}}(X_0^2 X_2 + X_1^2 X_3)$ (although it can be shown that this surface in fact has infinitely many lines). Indeed, if

$$L = \{[s : t : as + ct : bs + dt] \mid, s, t \in k\} \subseteq V^{\mathbb{P}}(X_0^2 X_2 + X_1^2 X_3)$$

then we must have

$$s^2(as + ct) + t^2(bs + dt) = as^3 + cs^2 t + bst^2 + dt^3 = 0 \qquad \text{for all } s, t \in k.$$

This is only possible if $a = b = c = d = 0$, that is, $L = L_0$. Thus we have shown $\pi_2^{-1}(X_0^2 X_2 + X_1^2 X_3) \cup U = \{L_0\}$. We conclude that L_0 is an isolated point in its fiber, that is, it is its own connected component, and thus L_0 is an irreducible component in the fiber $\pi_2^{-1}(X_0^2 X_2 + X_1^2 X_3)$. Thus the local dimension of $\pi_2^{-1}(\pi_2(L_0))$ at L_0 is 0. By the Theorem on Dimension of Fibers, this says that

$$19 - \dim Z = \dim \Sigma - \dim Z \leq \dim_{L_0} \pi_2^{-1}(\pi_2(L_0)) = 0.$$

Thus $\dim Z \geq 19$. But $Z \subseteq \mathbb{P}^{19}$ which is irreducible of $\dim \mathbb{P}^{19}_k = 19$, so by homework $Z = \mathbb{P}^{19}_k$ and thus π_2 is surjective. Moreover, the dimension of fibers theorem says that there is an open subset O of $\mathbb{P}^{19}$ so that for $F \in O$ we have

$$0 = \dim \pi_2^{-1}(F) = \dim\{L \mid L \subseteq V^{\mathbb{P}}(F)\}.$$

Since the only 0-dimensional varieties are finite sets of points, it follows that a general cubic surface contains finitely many lines.

Finally suppose that $F(X_0, X_1, X_2, X_3)$ is a homogeneous polynomial of degree $d \geq 4$. Then we have an analogous incidence correspondece

$$\Sigma = \{(L, F) \mid L \text{ line}, F \text{ homogeneous of degree d}, L \subseteq V^{\mathbb{P}}(F)\} \subseteq \mathrm{Gr}(2,4) \times \mathbb{P}^{\binom{d+3}{3}},$$

Adapting the argument involving π_1 from above gives

$$\dim \Sigma = 4 + \binom{d+3}{3} - (d+1) < \binom{d+3}{3}.$$

Since $\dim \Sigma < \dim \mathbb{P}^{\binom{d+3}{3}}$, by the Theorem on Dimension of Fibers the second projection cannot be dominant. Thus the complement of $\pi_2(\Sigma)$ is a nonempty open set W and each $F \in W$ has the property that $V^{\mathbb{P}}(F)$ contains no lines. $\square$

Chapter 4

Blow-ups

Monday, March 24

4.1 Definition and examples

Here is the prototypical example of a blow-up.

Example 4.1. Consider the incidence correspondence

$$B = \{(\mathbf{a}, L) \mid \mathbf{a} \in \mathbb{A}^n_k, L \text{ a line through } \mathbf{a} \text{ and the origin in } \mathbb{A}^n_k\} \subseteq \mathbb{A}^n_k \times \mathbb{P}^{n-1}_k.$$

Let the coordinate ring of $\mathbb{A}^n_k \times \mathbb{P}^{n-1}_k$ be $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$. We show that B is a Zariski closed subset of $\mathbb{A}^n_k \times \mathbb{P}^{n-1}_k$ and we find its defining equations. A point

$$\begin{aligned}
(\mathbf{a}, L) \in \mathbb{A}^n_k \times \mathbb{P}^{n-1}_k \text{ belongs to } B &\iff \\
\text{given } L = [\ell_1 : \ell_2 : \cdots : \ell_n] \text{ then } [\mathbf{a}] = [\ell_1 : \ell_2 : \cdots : \ell_n] &\iff \\
\text{the following matrix has rank one } \begin{bmatrix} \ell_1 & \ell_2 & \cdots & \ell_n \\ a_1 & a_2 & \cdots & a_n \end{bmatrix} &\iff \\
(\mathbf{a}, L) \in \mathbb{V}^{\mathbb{A}\times\mathbb{P}}(2 \times 2 \text{ minors of } \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ x_1 & x_2 & \cdots & x_n \end{bmatrix}) &\iff \\
(\mathbf{a}, L) \in \mathbb{V}^{\mathbb{A}\times\mathbb{P}}(x_i y_j - x_j y_i : 1 \le i, j \le n).&
\end{aligned}$$

Let's consider the projection $\pi : B \to \mathbb{A}^n_k$. The fiber $\pi^{-1}(\mathbf{a})$ is a singleton for $\mathbf{a} \neq 0$ is not the origin because it consists of the pair $\mathbf{a}$ and the unique line connecting $\mathbf{a}$ with the origin. However, the fiber $\pi^{-1}(0) = \{0\} \times \mathbb{P}^{n-1}_k := E$ consists of all the lines in $\mathbb{A}^n_k$, that is all of $\mathbb{P}^{n-1}_k$. Since this fiber is different, we refer to it as the *exceptional set*.

From this we see that π induces an isomorphism between the open sets $U = \mathbb{A}^n_k \setminus \{0\}$ and $U' = B \setminus E$ with inverse $\mathbf{a} \mapsto (\mathbf{a}, [\mathbf{a}])$. We will see below that since $\mathbb{A}^n_k$ and B have isomorphic dense open sets it follows that $\mathbb{A}^n_k$ and B are birational. However, they are not isomorphic.

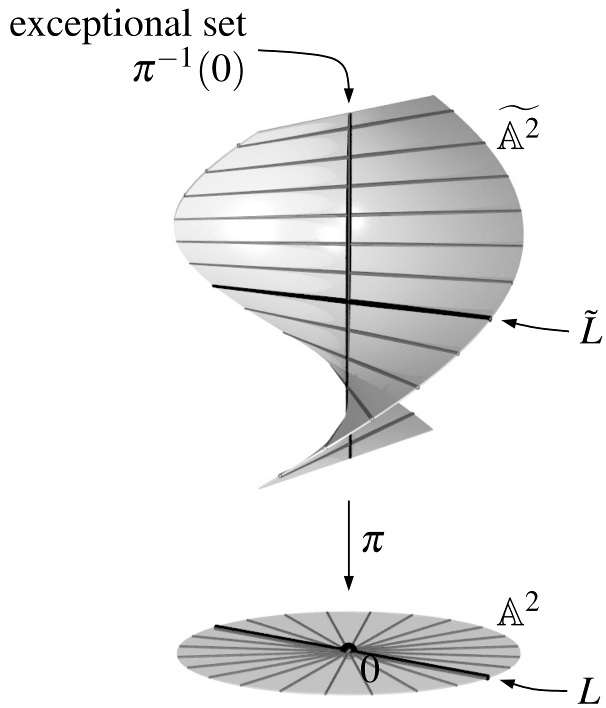


Figure 4.1: Blow-up at $\mathbb{A}^2_k$ at the origin

For the case $n = 2$ A picture of (a portion of) B and the projection is included below, where B is denoted $\widetilde{\mathbb{A}^2_k}$. The picture is missing a line at infinity. The complete picture of $\widetilde{\mathbb{A}^2_k}$ would look like a Möbius strip (but unbounded along the direction marked L): when the strip has rotated 360^o around the exceptional fiber, its end will be glued back to its beginning.

The variety B in our example above is the *blow-up of* $\mathbb{A}^n_k$ *at the origin.*

We now generalize this.

Definition 4.2. Let $X \subseteq \mathbb{A}^n_k$ be a classical affine variety and $f_0, \ldots, f_r$ elements of $k[X] = k[x_1, \ldots, x_n]/I(X)$ not all equal to 0 and set $U = X \setminus V(f_0, \ldots, f_r)$. We have a well-defined morphism of quasi-projective varieties

$$f : U \to \mathbb{P}^r,\ f(a_1, \ldots, a_n) = [f_0(a_1, \ldots, a_n) : f_1(a_1, \ldots, a_n) : \cdots : f_r(a_1, \ldots, a_n)]$$

The graph of this morphism is the closed sub-variety of $U \times \mathbb{P}^r$

$$\Gamma_f := \{(\mathbf{a}, f(\mathbf{a})) \mid \mathbf{a} \in U\} \subseteq U \times \mathbb{P}^r.$$

Since Γ_f is closed in $U \times \mathbb{P}^r$ but $U \times \mathbb{P}^r$ is open in $X \times \mathbb{P}^r$, we have that (except in trivial situations) Γ_f is not closed in $X \times \mathbb{P}^r$.

The *blow-up of* X *along* $f_0, \ldots, f_r$ is $\widetilde{X}_{f_0,\ldots,f_r} = \overline{\Gamma_f}$, the Zariski closure of Γ_f in $X \times \mathbb{P}^r$. Note that in general, $X \times \mathbb{P}^r$ is neither affine nor projective, only quasi-projective, thus $\widetilde{X}$ is also in general only a quasi-projective variety. We write this quasi-projective variety $\widetilde{X}_f$ for short or sometimes as $\tilde{X}$ when the f_i's are clear from context.

To talk about Zariski closure in $X \times \mathbb{P}^r_k \subseteq \mathbb{A}^n \times \mathbb{P}^r_k$ we need to know what are the Zariski closed subsets of $\mathbb{A}^n_k \times \mathbb{P}^r_k$. Let $k[x_1, \dots, x_n]$ be the affine coordinate ring of $\mathbb{A}^n_k$ and $k[Y_0, \dots, Y_r]$ the projective coordinate ring of $\mathbb{P}^r_k$. Declare an element of $k[x_1, \dots, x_n, Y_0, \dots, Y_r]$ to be homogenous of degree d if it has the form $\sum_I f_I(x_1, \dots, x_n) Y^I$ where I ranges over $r+1$ tuples with $|I| = d$. In other words, make $k[x_1, \dots, x_n; Y_0, \dots, Y_r]$ into a graded ring by declaring $\deg(x_i) = 0$ and $\deg(Y_j) = 1$.

Proposition 4.3. *For any homogeneous elements $F_1, \dots, F_l$ of the graded ring $k[x_1, \dots, x_n, Y_0, \dots, Y_r]$, where $\deg(x_i) = 0$ and $\deg(Y_j) = 1$, the set*

$$V^{\mathbb{A}\times\mathbb{P}}(F_1, \dots, F_l) := \{((a_1 \dots, a_n), [b_0 : b_1 : \dots : b_r]) \in \mathbb{A}^n \times \mathbb{P}^r \mid F_j(a_1, \dots, a_n, b_0, \dots, b_r) = 0, \forall j\}.$$

is a well-defined, closed subset of $\mathbb{A}^n \times \mathbb{P}^r$ and every closed subset of $\mathbb{A}^n \times \mathbb{P}^r$ has this form.

With the notation in definition 4.2, let $\pi : \tilde{X}_f \to X$ be the morphism induced by the projection $X \times \mathbb{P}^r \to X$. Since $\tilde{X}_f \cap U \times \mathbb{P}^r = \Gamma_f$ the fibers of π over a point $P \in U$ is the one-point set $(U, f(U))$. In fact, $\pi^{-1}(U) = \Gamma_f \subseteq U \times \mathbb{P}^r$. But for a point P outside of U, the fiber will often be a closed subset of $\{P\} \times \mathbb{P}^r \cong \mathbb{P}^r$ typically consisting of more than one point.

We now see that $U \cong \Gamma_f$ and that the blow-up $\widetilde{X}_f$ acts as a domain that the map f can be extended to. Moreover we can compute the blow-up of a variety Z by first blowing up the ambient space X and then taking the *stric transform* of Z, that is, the closure of $\pi^{-1}(Z)$ withe the exceptional set removed.

Proposition 4.4 (Properties of blow-ups)**.** *For the blow-up $\widetilde{X}_f$ in Definition 4.2, setting $Y = \mathbb{V}^{\mathbb{A}}(f_0, \dots, f_r)$ we have*

1. *the projection $\pi : \widetilde{X}_f \to X$ induces an isomorphism $X \setminus Y = U \cong \Gamma_f = \widetilde{X}_f \setminus \pi^{-1}(Y)$.*

2. *if X is irreducible, then $\dim \widetilde{X}_f = \dim X$*

3. *the regular map f extends to a map $\widetilde{X}_f \to \mathbb{P}^r_k$.*

4. *if $Z \subseteq X$ is a subvariety such that $Y \subseteq Z \subseteq X$, then $\widetilde{Z} \subseteq \widetilde{X}$ and $\widetilde{Z} = \overline{\pi^{-1}(Z) \setminus \pi^{-1}(Y)}$, where $^-$ denotes Zariski closure in $\widetilde{X}$ and $\pi : \widetilde{X} \to X$ is the projection map. The sub-variety $\widetilde{Z}$ is called the* strict transform *of Z in $\widetilde{X}$.*

Proof. 1. The inverse map is the regular map $\iota : U \to \widetilde{X}_f, \iota(\mathbf{a}) = (\mathbf{a}, f(\mathbf{a}))$.

2. Since the f_i are not all zero, we know that $(f_0, \dots, f_r) \neq (0)$ thus $Y \neq X$. Then U is dense in X and Γ_f is dense in $\widetilde{X}_f$ so by part 1. we have $\dim X = \dim U = \dim \Gamma_f = \dim \widetilde{X}_f$.

3. We have that f factors as $U \xrightarrow{\iota} \widetilde{X}_f \xrightarrow{\pi_2} \mathbb{P}^r_k$. Identifying U with $\iota(U)$ gives that $\pi_2 : \widetilde{X}_f \to \mathbb{P}^r_k$ extends f.

4. By definition, if Γ_f is the graph of $f : X \setminus Y \to \mathbb{P}^r_k$ and Γ'_f is the graph of $f : Z \setminus Y \to \mathbb{P}^r_k$, then $\Gamma'_f \subseteq \Gamma_f$. Taking Zariski closures gives $\widetilde{Z} \subseteq \widetilde{X}$.

It is clear that $\Gamma'_f \subseteq \pi^{-1}(Z)$, and since $\pi^{-1}(Z)$ is Zariski closed we conclude that $\widetilde{Z} \subseteq \pi^{-1}(Z)$. By part 1. there is an isomorphism $Z \setminus Y \cong \pi^{-1}(Z) \setminus \pi^{-1}(Y) = \Gamma'_f$, whence the claim follows. □

Wednesday, March 26

We now compute several examples of blow-ups.

Example 4.5. Let $X = \mathbb{A}^2$ with coordinate ring $k[x, y]$, $r = 1$, and $f_0 = x$ and $f_1 = y$. Then $U = X \setminus \{(0, 0)\}$ and recall that we can think of

$$f : U \to \mathbb{P}^1,\ f(a, b) = [a : b]$$

as the function sending a point in the plane to the slope of the line joining that point and the origin. (A point $[a : b] \in \mathbb{P}^1$ is identified with the slope b/a, interpreted as the slope of a vertical line when $a = 0$.)

I claim $\widetilde{\mathbb{A}^2} = \widetilde{\mathbb{A}^2}_{x,y}$ is the closed subset Z of $\mathbb{A}^2 \times \mathbb{P}^1$ given by

$$Z := V^{\mathbb{A}\times\mathbb{P}}(xT - yS) = \{(a, b), [c : d]) \mid ad = bc\} \subseteq \mathbb{A}^2 \times \mathbb{P}^1$$

where we write the projective coordinate ring of $\mathbb{P}^1$ as $k[S, T]$.

Let us prove this carefully: If we remove $\pi^{-1}(0, 0)$ (i.e., the vertical line in the picture) from Z we get the set

$$\{(a, b), [c : d]) \mid (a, b) \neq (0, 0), ad = bc\} = \{(a, b), [a : b]) \mid (a, b) \neq (0, 0)\}$$

which is precisely the graph Γ_f of $f : U \to \mathbb{P}^1$. Thus $Z = \Gamma_f \cup \{(0, 0)\} \times \mathbb{P}^1$. Since Z is closed in $\mathbb{A}^2 \times \mathbb{P}^1$, it follows that $\widetilde{\mathbb{A}^2} \subseteq Z$. On the other hand, suppose $C \subseteq \mathbb{A}^2 \times \mathbb{P}^1$ is any closed subset that contains Γ_f. We need to show C contains $\{(0, 0)\} \times \mathbb{P}^1$. For any fixed point $[c : d] \in \mathbb{P}^1$, we have

$$\Gamma_f \cap (\mathbb{A}^2 \times \{[c : d]\}) = \{((a, b), [c : d]) \mid (a, b) \neq (0, 0), ad = bc\}$$

which we may identify with the subset $L \setminus \{(0, 0)\}$ of $\mathbb{A}^2$, where L is the line in $\mathbb{A}^2$ containing the origin and (c, d). Note that the closure of $L \setminus \{(0, 0)\}$ in $\mathbb{A}^2$ is all of L and hence the closure of $\Gamma_f \cap \mathbb{A}^2 \times \{[c : d]\}$ in $\mathbb{A}^2 \times \{[c : d]\}$ contains $(0, 0, [c : d])$. Since $C \cap \mathbb{A}^2 \times \{[c : d]\}$ is closed in $\mathbb{A}^2 \times \{[c : d]\}$, we conclude that C must also contain $(0, 0, [c : d])$. Since $[c : d]$ was arbitrary, this proves C contains $\{(0, 0)\} \times \mathbb{P}^1$.

So $\widetilde{\mathbb{A}^2} = Z$, which is pictured above in Figure 4.1. A bit of intuition the image above suggests: by blowing up the origin in $\mathbb{A}^2_k$, we have replaced the origin with $\mathbb{P}^1$ and in this way we keep track of how we approach the origin: If we approach the origin along the line $L : xb - ya = 0$, the lift of line to $\widetilde{\mathbb{A}^2}$ intersects the copy of $\mathbb{P}^1$ at $[a : b]$, the slope of this line, interpreted as a point in $\mathbb{P}^1$.

Let us reinforce the intuition regarding tangent directions with a singular example:

Example 4.6. Take X to be the affine plane curve $X = V(y^2 - x^2(x+1))$, the classical node, and let $f_0 = x$ and $f_1 = y$ as before (but now interpreted as elements in $k[X]$.) Then I claim that the blowup $\tilde{X}$ along f_0, f_1 is the subset of $X \times \mathbb{P}^1 \subseteq \mathbb{A}^2 \times \mathbb{P}^1$ given by

$$\widetilde{X} = C := V^{\mathbb{A}^2 \times \mathbb{P}^1}(y^2 - x^2(x+1), xT - yS, T^2 - S^2(x+1))$$

where as before $k^{\mathbb{P}}[\mathbb{P}^1] = k[S,T]$. How did I come up with this? The first two equations are clear. Starting with $y^2 = x^2(x+1)$ the second equation gives $x = y(S/T)$ (provided $T \neq 0$) and thus $y^2 = y^2(S^2/T^2)(y(S/T)+1)$ and so upon multiplying through by T^3 we get $T^3y^2 = y^2S^2(yS+T) = y^2S^2(xT+T)$, and now divide through by the common factor of y^2T to get $T^2 = S^2(x+1)$. One would get the same conclusion if one assumed that $S \neq 0$ and solved for $y = x(T/S)$. Since at least one of S or T are not zero, this reasoning says that the third equation must hold on the blow-up $\widetilde{X}$.

Let us now check carefully that $\widetilde{X} = C$:

First we note that $\Gamma_f \subseteq C$. Since C is closed in $\mathbb{A}^2 \times \mathbb{P}^1$, it follows that $\widetilde{X}_f \subseteq C$. To show equality, we just need to show $\widetilde{X}_f$ is dense in C. To accomplish that, we first show that C is isomorphic to the affine line:

Notice the last polynomial in the definition of C gives that if $S = 0$ then $T = 0$, and hence $C \cap \mathbb{A}^2 \times \{[0:1]\} = \emptyset$. Thus C is a closed subset of $\mathbb{A}^2 \times U_1 \subseteq \mathbb{A}^2 \times \mathbb{P}^1$. Under the isomorphism $\mathbb{A}^3 \cong \mathbb{A}^2 \times U_1$ ("set $S = 1$ and replace T with $t = T/S$"), we have

$$C \cong V(y^2 - x^2(x+1), xt - y, t^2 - (x+1)) \subseteq \mathbb{A}^3$$

where $k[\mathbb{A}^3] = k[x,y,t]$. The last equation allows us to solve for x in terms of t and then the second allows us to solve for y in terms of t. Moreover, the first equation is a consequence of the latter two. So:

$$V(y^2 - x^2(x+1), xt - y, t^2 - (x+1)) = V(x - t^2 + 1, y - y^3 + t).$$

In terms of coordinate rings we have

$$\begin{aligned} k[x,y,t]/\left(y^2 - x^2(x+1), xt - y, t^2 - (x+1)\right) &\cong k[x,t]/\left((xt)^2 - x^2(x+1), t^2 - (x+1)\right) \\ &\cong k[t]/\left(((t^2-1)t)^2 - (t^2-1)^2t^2\right) \\ &= k[t]. \end{aligned}$$

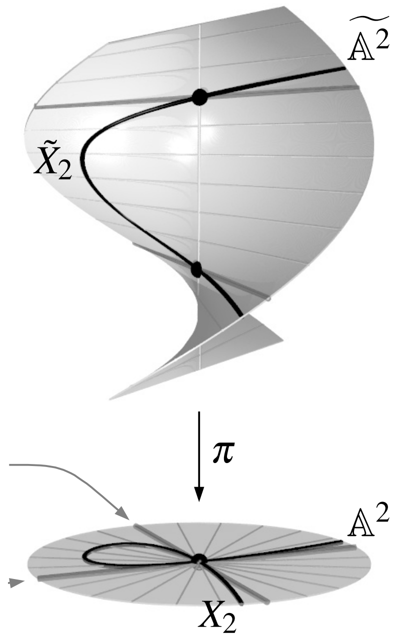
This proves $\mathbb{A}^1 \cong V(x - t^2 + 1, y - y^3 + t)$ via the map sending z to $(z^2 - 1, z^3 - z)$. In other words, we have an isomorphism $g : \mathbb{A}^1 \xrightarrow{\cong} C$ given by

$$g(z) = (z^2 - 1, z^3 - z, [1 : z]).$$

(Note that $[z^2 - 1 : z^3 - z] = [1 : z]$ provided $z^2 - 1 \neq 0$.) Finally, under this isomorphism, Γ_f corresponds to $\mathbb{A}^1 \setminus \{1, -1\}$. This is clearly dense in $\mathbb{A}^1$ and thus Γ_f is dense in C. Hence, $\tilde{X} = C$ as claimed.

We note that the composition of the isomorphism $\mathbb{A}^1 \xrightarrow{\cong} \widetilde{X}$ and the canonical map $\pi : \widetilde{X} \to X$ is the parameterization of X that we worked out before. Recall this parameterization was onto but not injective. By blowing up, we've made it into an isomorphism.

Here is a picture of C from Gathmann:



Taking it for granted that $\widetilde{X} = C$, we see that the fiber of $\pi : \widetilde{X} \to \mathbb{A}^2$ over any point other than the origin is just one point and the fiber over $(0, 0)$ may be identified with $V^{\mathbb{P}}(T^2 - S^2) = \{[1 : 1], [1 : -1]\} \subseteq \mathbb{P}^1$. Note that the "two tangent lines" to X at the origin have been "pulled apart" in the blow-up $\widetilde{X}$, and indeed as shown above $\widetilde{X}$ becomes smooth (in fact, as we shall soon see, it is isomorphic to $\mathbb{A}^1$).

Example 4.7. Let us play the same game starting with cuspidal affine plane curve $X = V(y^2 - x^3)$. Let $\widetilde{X}$ be the blow-up of X along x, y so that

$$\widetilde{X} \subseteq V(y^2 - x^3, xT - yS) \subseteq \mathbb{A}^2 \times \mathbb{P}^1.$$

Note that if $y^2 = x^3$ then $(y/x)^2 = x$ for all $x \neq 0$. Given $xT = yS$, we have $y/x = T/S$ (if $S \neq 0$) and thus $(T/S)^2 = x$ and hence $T^2 = xS^2$. This suggests that

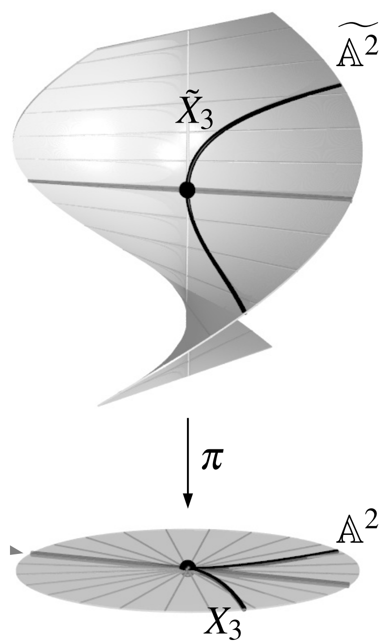
$$\widetilde{X} = Z := V(y^2 - x^3, xT - yS, T^2 - xS^2) \subseteq \mathbb{A}^2 \times \mathbb{P}^1.$$

Indeed, since $\Gamma_f = \{(a, b, [a : b] \mid (a, b) \neq (0, 0), b^2 = a^3\} \subseteq Z$, $\widetilde{X}_f$ is the closure of Γ_f and Z is closed, we certainly have $\widetilde{X} \subseteq Z$. To show the opposite containment, we first prove $Z \cong \mathbb{A}^1$:

To see this, first note that $Z \cap \mathbb{A}^2 \times \{[0:1]\} = \emptyset$ (since $V(T^2 - xS^2) \cap \mathbb{A}^2 \times \{[0:1]\} = \emptyset$). So Z is a closed sub-variety of $\mathbb{A}^2 \times U_1 \subseteq \mathbb{A}^2 \times \mathbb{P}^1$. Moreover, under the isomorphism $\mathbb{A}^3 \cong \mathbb{A}^2 \times U_1$, Z corresponds to $V(y^2 - x^3, xt - y, t^2 - x)$ where $k[\mathbb{A}^3] = k[x, y, t]$. We have $V(y^2 - x^3, xt - y, t^2 - x) = V(xt - y, t^2 - x) = V(y - t^3, x - t^2)$ and this shows that the map $\mathbb{A}^1 \xrightarrow{\cong} V(y^2 - x^3, xt - y, t^2 - x)$ sending a to (a^2, a^3) is an isomorphism. In other words, we have an isomorphism $\mathbb{A}^1 \xrightarrow{\cong} Z$ sending a to $(a^2, a^3, [1:a])$ is an isomorphism. (Note that $[1:a] = [a^2 : a^3]$ if $a \neq 0$.)

Now, under the isomorphism $\mathbb{A}^1 \cong Z$ constructed here, the subset Γ_f of Z corresponds to $\mathbb{A}^1 \setminus \{0\}$. Since $\mathbb{A}^1 \setminus \{0\}$ is dense in $\mathbb{A}^1$, Γ_f is dense in Z. Thus $\tilde{X}_f = Z$.

Here is a picture:



This time there is just one tangent line to X and the origin, and this is why there is just one point in $\tilde{X}$ lying over the origin.

As before the composition of $\mathbb{A}^1 \xrightarrow{\cong} \widetilde{X} \xrightarrow{\pi} X$ is the parameterization of X that we knew about before. Recall this parameterization is bijective but not an isomorphism of varieties.

Friday, March 28

4.2 The coordinate ring of the blow-up

4.2.1 Properties of blow-ups

Let us record some general properties of blow-ups:

Proposition 4.8. *Assume $X \subseteq \mathbb{A}^n$ is a classical affine variety, let $f_0, \dots, f_r \in k[X]$, and set $\widetilde{X} \subseteq \mathbb{A}^n \times \mathbb{P}^r$ the blow-up of X along $f_0, \dots, f_r$. Let $\pi : \widetilde{X} \to X$ be the canonical map, let $x_1, \dots, x_n$ be affine coordinate functions for $\mathbb{A}^n$ and let $Y_0 \dots, Y_r$ be projective coordinate functions for $\mathbb{P}^r$.*

1. *We have $\widetilde{X} \subseteq V_X(\{f_iY_j - f_jY_i \mid i \neq j\}) \subseteq X \times \mathbb{P}^r$, but equality does not hold in general.*
2. *The equality $\tilde{X} = V_X(\{f_iY_j - f_jY_i \mid i \neq j\})$ does hold provided X is equidimensional and $f_0, \dots, f_r$ is a regular sequence, by which I mean that $\dim(V_X(f_0, \dots, f_r)) = \dim(X) - r - 1$ (the smallest possible value).*
3. *$\widetilde{X}$ depends only on the ideal of $k[X]$ generated by the f_i's up to isomorphism. That is, If $(g_0, \dots, g_s) = (f_0, \dots, f_r)$ and $\widetilde{X}'$ is the blowup of X along $g_0, \dots, g_s$, then there is an isomorphism $g : \widetilde{X} \to \widetilde{X}'$ such that $\pi' \circ g = \pi$.*

Proof.

1. For any point $P \in U$, the point $(P, [f_0(P) : \cdots : f_r(P)])$ in Γ_f is a zero of $f_iX_j - f_jX_i$ for all $i \neq j$ since $f_i(P)f_j(P) - f_j(P)f_i(P) = 0$. So $V_X(\{f_iY_j - f_jY_i \mid i \neq j\})$ contains Γ_f and, since it is closed in $X \times \mathbb{P}^r$ and $\widetilde{X}$ is the closure of Γ_f, it contains $\widetilde{X}$.
2. Let $Z = V_X(\{f_iY_j - f_jY_i \mid i \neq j\})$. Since $\Gamma_f \subseteq \widetilde{X}_f \subseteq Z$, $\widetilde{X}_f$ is the closure of Γ_f, and Z is closed in $X \times \mathbb{P}^r$, it suffices to prove that for each fixed point $[\mathbf{b}] = [b_0 : \cdots : b_r] \in \mathbb{P}^r$, we have that $\Gamma_f \cap X \times \{[\mathbf{b}]\}$ is dense in $Z \cap X \times \{[\mathbf{b}]\}$. We will identity both of these intersections with subsets of X using $X \cong X \times \{[\mathbf{b}]\}$.

   Let us prove this first when $\vec{b} = (1, 0, \dots, 0)$. In this case $Z \cap X \times \{[\vec{b}]\}$ corresponds to the closed subset $V_X(f_1, \dots, f_r)$ of X and $\Gamma_f \cap X \times \{[\vec{b}]\}$ corresponds to $\{P \in X | f_0(P) \neq 0, f_1(P) = \cdots = f_r(P) = 0\} = V_X(f_1, \dots, f_r) \setminus V_X(f_0)$. Recall that by Krull's Theorem, each irreducible component of $V_X(f_1, \dots, f_r)$ has dimension at least $\dim(X) - r$. Since we assume $V_X(f_0, \dots, f_r)$ has dimension equal to $\dim(X) - r - 1$, it follows that $V_X(f_0)$ must not contain any of the irreducible components of $V_X(f_1, \dots, f_r)$. This is equivalent to the statement that $V_X(f_1, \dots, f_r) \setminus V_X(f_0)$ is dense in $V_X(f_1, \dots, f_r)$.

   For the general case, there is an $(r+1) \times (r+1)$ invertible matrix A such that $(1, 0, \dots, 0) = \mathbf{b} \cdot A$. Set $(f'_0, \dots, f'_r) = (f_0, \dots, f_r)A$. The matrix A determines an automorphism of $\mathbb{P}^r_k$ and hence of $X \times \mathbb{P}^r$. Under it, the $X \times \{[\mathbf{b}]\}$ is sent to $X \times \{[1 : 0 : \cdots : 0]\}$, Γ_f is sent to $\Gamma_{f'}$ and Z is sent to $Z' = V_X(\{f'_iY_j - f'_jY_i \mid i \neq j\})$. Morever, it is not hard to see that $(f'_0, \dots, f'_r) = (f_0, \dots, f_r)$ and hence $\dim(V_X(f'_0, \dots, f'_r)) = \dim(V_X(f_0, \dots, f_r)) = \dim(X) - r - 1$. So, we have reduced to the case already considered.
3. (I'll just provide a sketch of a proof for this part.) For each $0 \leq i \leq s$ we have $g_i = \sum_{j=0}^{r} h_{i,j} f_j$ for some $h_{i,j} \in k[X]$. Consider the function $\phi : \widetilde{X} \to \widetilde{X}'$ given

by sending

$$\phi(\mathbf{a}, [a_0 : \cdots : a_r])) = (\mathbf{a}, [\sum_j h_{0,j}(\mathbf{a})a_j : \sum_j h_{1,j}(\mathbf{a})a_j : \cdots : \sum_j h_{s,j}(\mathbf{a})a_j]).$$

It is well-defined and a morphism of varieties. Likewise, we have $f_i = \sum_j h'_{i,j} g_j$ which leads to a morphism $\phi' : \widetilde{X}' \to \widetilde{X}$. These are mutually inverse.

□

Remark 4.9. In view of part 3 of the Theorem above henceforth we will refer to the blow-up of an affine variety X along an *ideal* $I = (f_0, \ldots, f_r)$ of $k[X]$, and write it as $\widetilde{X}_I$.

Monday, March 31 & Wednesday, April 2

4.2.2 The Rees algebra of an ideal

Definition 4.10. Let R be a ring and I be an ideal. The *Rees ring* of I is the $\mathbb{N}$-graded R-algebra

$$\mathcal{R}(I) = R[IT] := \bigoplus_{d \geq 0} I^d T^d = R \oplus IT \oplus I^2T^2 \oplus \cdots \subseteq R[T]$$

with multiplication determined by $(aT^d)(bT^e) = abT^{d+e}$ for $a \in I^d, b \in I^e$ and extended by the distributive law for nonhomogeneous elements and grading determined by $\deg(T) = 1$ and $\deg(r) = 0$ for $r \in R$.

Here T is an indeterminate and I^d means the d-th power of the ideal I in R, namely the ideal generated by d-fold products of elements of I

$$I^d = (u_1 u_2 \cdots u_r \mid u_i \in I).$$

If $I = (f_0, \ldots, f_r)$ and we wish to find a presentation of $\mathcal{R}(I)$ we may start by recalling that $\mathcal{R}(I)$ is generated as a R-algebra by $f_0T, \cdots, f_rT$. Thus we have a surjective map of R-algebras

$$\varphi : R[Y_0, \ldots, Y_r] \twoheadrightarrow \mathcal{R}(I) \qquad \varphi(Y_i) = f_iT. \tag{4.1}$$

Let P denote the kernel of φ. By the first isomorphism theorem, $\mathcal{R}(I) \cong R[Y_0, \ldots, Y_r]/P$ is a presentation of the Rees algebra. We now determine P.

Proposition 4.11. *Suppose R is a domain and $I = (f_0, \ldots, f_r)$ with each $f_i \neq 0$ is an ideal of R. Then Setting $S = R[Y_0, \ldots, Y_r]$ and $L = (f_iY_j - f_jY_i) \subseteq S$ we have*

$$\mathcal{R}(I) \cong R[Y_0, \ldots, Y_r]/P \text{ where } P = L : IS^\infty.$$

To prove this we need a lemma.

Lemma 4.12. *Suppose that P is a prime ideal in a ring S. Suppose that I, L are ideals in S such that $I \not\subset P, L \subset P$ and the localizations $L_Q = P_Q$ for any prime ideal Q in S such that $I \not\subset Q$. Then*

$$P = L :_S I^\infty := \{f \in S \mid fI^d \subseteq L \text{ for some } d \geq 0\}.$$

Proof. L has primary decomposition $L = Q_1 \cap \cdots \cap Q_s \cap P$ with $I \subseteq \sqrt{Q_i}$ for all $1 \leq i \leq s$. Thus there exists $d \gg 0$ such that $I^d \cap Q_i$ for all $1 \leq i \leq s$ yieldsing $I^d P \subseteq L$ and therefore $P \subseteq L : I^\infty$.

Conversely, if $f \in L : I^\infty$ we have $fI^d \subseteq L \subseteq P$. Since there exists $g \in I^d$ such that $g \notin P$ and P is prime it follows that $f \in P$, so $L : I^\infty \subseteq P$. □

Proof of Theorem 4.11. Setting $S = R[Y_0, \dots Y_r]$ and P to be the kernel of the map (4.1), we see P is a prime ideal since $\mathcal{R}(I) \subseteq R[T]$ is a domain. We prove the claimed description $P = L :_S I^\infty$ by localization.

Notice that $L \subseteq P$. We show that for $0 \leq i \leq r$ we have $LS[1/f_i] = PS[1/f_i]$.

Indeed, for each i, $LS[1/f_i] = (Y_j - \frac{f_j}{f_i} Y_i \mid 0 \leq j \leq r)$. Let $F \in P$ have degree d. Then we have

$$F \equiv rY_i^d \pmod{LS[1/f_i]}, \text{ for some } r \in R[1/f_i].$$

Clearing denominators we may assume $r \in R$. Since $F \in P$ we have $0 = \varphi(F) = rf_i^d$ in $\mathcal{R}(I)$. Since $\mathcal{R}(I)$ is a domain and $f_i \neq 0$ then $r = 0$. Thus we have shown $LS[1/f_i] = PS[1/f_i]$.

If Q is a prime ideal in S so that $I \nsubseteq Q$ then there is some $f_i \notin Q$. Then

$$L_Q = LS[1/f_i]_Q = PS[1/f_i]_Q = P_Q.$$

Finally we show that $I \nsubseteq P$. Suppose that $I \subseteq P$ then by the above $S[1/f_i] = IS[1/f_i] \subseteq PS[1/f_i]$ gives $PS[1/f_i] = LS[1/f_i] = S[1/f_i]$. The equation (4.1) then gives that $\mathcal{R}(I)[1/f_i] = 0$, which is a contradiction since $f_i \neq 0$ and $\mathcal{R}(I)$ is a domain.

From all of the above the claimed conclusion follows by Lemma 4.12. □

Example 4.13. If I is an ideal generated by a regular sequence $f_0, \dots, f_r$ then

$$\mathcal{R}(I) \cong R[Y_0, \dots, Y_r]/(f_i Y_j - f_j Y_i).$$

This follows from Proposition 4.8 and Theorem 4.14.

4.2.3 The coordinate ring of a blow-up

Recall that in the set-up of blow-ups the blow-up of a variety X is a a subset $\widetilde{X} \subseteq X \times \mathbb{P}_k^r$. We know that the coordinate ring of $X \times \mathbb{P}_k^r$ is $S = k[X] \otimes_k [Y_0, \dots Y_r] = k[X][Y_0, \dots, Y_r]$. We wish to find the defining ideal of the blow-up, that is,

$$I_{X \times \mathbb{P}_k^r}(\widetilde{X}_f) := (g \in k[X][Y_0, \dots, Y_r] \mid g(\mathbf{a}, f(\mathbf{a})) = 0, \ \forall (\mathbf{a}, f(\mathbf{a})) \in \Gamma_f).$$

and we define the coordinate ring of $\widetilde{X}$ by $k[\widetilde{X}] = S/I_{X \times \mathbb{P}_k^n}(\widetilde{X})$.

Our main goal is to prove the following:

Theorem 4.14. *Suppose X is an irreducible affine variety and $I = (f_0, \dots, f_r) \subseteq k[X]$ is an ideal with all $f_i \neq 0$ and $\widetilde{X} = \widetilde{X}_I \subseteq X \times \mathbb{P}^r_k$ has coordinate ring $S = k[X][Y_0, \dots Y_r]$. Then the coordinate ring of the blow-up is isomorphic to the Rees ring of I as graded R-algebras*

$$k[\widetilde{X}] = \frac{S}{I_{X \times \mathbb{P}^n_k}(\widetilde{X})} \cong \mathcal{R}(I).$$

In particular, if $L = (f_iY_j - f_jY_i \mid 0 \leq i, j \leq r)$ then

$$I(\widetilde{X}) = L :_S IS^\infty = \{f \in S \mid fI^d \subseteq L \text{ for some } d \geq 0\}.$$

Example 4.15. Consider the cusp $X = V^{\mathbb{A}}(y^2 - x^3)$ and $I = (x, y) \subset k[X]$. Then $L = (xT - yS) \subseteq \frac{k[x,y,S,T]}{(y^2-x^3)}$ corresponds to the ideal $L' = (y^2 - x^3, xT - yS)$ of $k[x, y, S, T]$ by the lattice isomorphism theorem. Then

$$L' : I^\infty = (y^2 - x^3, xT - yS, yT - x^2S, T^2 - xS^2)$$

corresponds by lattice isomorphism to

$$L : I^\infty = (xT - yS, yT - x^2S, T^2 - xS^2) \subseteq \frac{k[x, y, S, T]}{(y^2 - x^3)}$$

and thus

$$k[\widetilde{X}] \cong \frac{k[x, y, S, T]}{(y^2 - x^3, xT - yS, yT - x^2S, T^2 - xS^2)}.$$

Remark 4.16. The defining ideal of a blow-up at I encodes relations among the generators of the powers of I. In the example above with $I = (f_0 = x, f_1 = y)$ the equations of the blow-up translates as follows

$xT - yS$	is mapped to the "Koszul syzygy"	$xf_1 - yf_0 = 0$
$yT - x^2S$	is mapped to the relation	$yf_1 - x^2f_0 = 0$
$T^2 - xS^2$	is mapped to the relation on I^2	$f_1^2 - xf_0^2 = 0.$

We now prove Theorem 4.14 using Proposition 4.11.

Proof of Theorem 4.14. We have an inclusion $\iota : \widetilde{X}_I \subseteq X \times \mathbb{P}^r_k$ which induces a map between the coordinate rings

$$\iota^* : S := k[X \times \mathbb{P}^r_k] \to k[\widetilde{X}_I].$$

This map is surjective by definition of $k[\widetilde{X}_I]$ (every function in $k[\widetilde{X}_I]$ is a restriction of a function in S to $\widetilde{X}$).

I claim this map is the same as in (4.1), namely it is given by $\iota^*(Y_i) = f_i$ and $\iota^*(r) = r$ for each $r \in R = k[X]$. By definition of pullback, $\iota^*(Y_i) = Y_i \circ \iota = f_i$ at least on Γ_f since $Y_i \circ \iota$ picks out the i-th coordinate of $[f_0(\mathbf{a}) : \cdots : f_r(\mathbf{a})]$. Since $Y_i \circ \iota = f_i$ is true on Γ_f it is also true on its closure $\widetilde{X}_I$. The proof for $\iota^*(r) = r \circ \iota = r$ is similar.

It follows that $k[\widetilde{X}_I] \cong \mathcal{R}(I)$.

□

Chapter 5

Smoothness and Tangent Space

Friday, April 4 2025

5.1 Tangent space and tangent cone

We study two ways to approximate a given variety around a given point: one approximation is given by a linear space (the tangent space) and another by an affine cone (the tangent cone). Since both of these are defined "at a point", they are local properties of quasi-projective varieties. Since locally these varieties are affine we will restrict to the case of affine varieties right away.

5.1.1 Tangent space

Every polynomial in $k[x_1, \ldots, x_n]$ has a Taylor expansion at $\mathbf{a} \in \mathbb{A}_k^n$ given by

$$f(x_1, \ldots, x_n) = f(\mathbf{a}) + \underbrace{\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{a})(x_i - a_i)}_{f_{1,\mathbf{a}}} + \underbrace{\frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i \partial x_j}(\mathbf{a})(x_i - a_i)(x_j - a_j) + \cdots}_{\in I(\mathbf{a})^2}$$

We define $f_{1,\mathbf{a}}$ to be the homogeneous term of degree 1 in this expansion as indicated above.

Definition 5.1. Let $\mathbf{a}$ be a point on an affine variety $X \subseteq \mathbb{A}_k^n$. Then the *tangent space* of X at $\mathbf{a}$ is the linear space

$$T_{\mathbf{a}}X := V^{\mathbb{A}}(f_{1,\mathbf{a}} \mid f \in I(X)) \subseteq A_k^n.$$

Here is a more concrete way of computing this space.

Lemma 5.2. *In the definition above it suffices to consider generators, that is, if* $I = (f^{(1)}, \ldots, f^{(s)})$ *then*

$$T_{\mathbf{a}}X := V^{\mathbb{A}}(f_{1,\mathbf{a}}^{(j)} \mid 1 \leq j \leq s).$$

there is an isomorphism

$$T_{\mathbf{a}}X \cong V^{\mathbb{A}}\left(\frac{\partial f^{(j)}}{\partial x_i}(\mathbf{a})y_i\right) = \mathrm{Null}\left(\frac{\partial f^{(j)}}{\partial x_i}(\mathbf{a})\right).$$

Proof. The isomorphism is given by $x_i - a_i \mapsto y_i$, that is, translation by $\mathbf{a}$. □

Example 5.3. The tangent space of the parabola $X_1 = V^{\mathbb{A}}(y - x^2)$ at the origin is the horizontal axis $T_0X_1 = V^{\mathbb{A}}(y)$ while the tangent space of both $X_2 = V^{\mathbb{A}}(y^2 - x^2(x+1))$ and $X_3 = V^{\mathbb{A}}(y^2 - x^3)$ at the origin is $T_0X = V(0) = \mathbb{A}^2_k$.

We can give an intrinsic definition for the tangent space that does not depend on the precise defining equations of X but rather only local information regarding the point $\mathbf{a} \in X$.

Recall that the regular functions at $\mathbf{a}$ on X form a local ring $\mathcal{O}_{X,\mathbf{a}} = k[X]_{I(\mathbf{a})}$ by Corollary 1.90. This ring is invariant with respect to isomorphism, i.e., if $f : X \to Y$ is an isomorphism, then $\mathcal{O}_{X,\mathbf{a}} \cong \mathcal{O}_{Y,f(\mathbf{a})}$.

Theorem 5.4. *Let $I_X(\mathbf{a})$ denote the ideal of $k[X]$ defining $\mathbf{a}$ and $\mathfrak{m}_{\mathbf{a}} = I_X(\mathbf{a})\mathcal{O}_{X,\mathbf{a}}$. There is a natural vector space isomorphism $\mathfrak{m}_{\mathbf{a}}/\mathfrak{m}_{\mathbf{a}}^2 \cong I_X(\mathbf{a})/I_X(\mathbf{a})^2 \cong \mathrm{Hom}_k(T_{\mathbf{a}}X, k)$. In other words, the tangent space $T_{\mathbf{a}}X \cong \mathrm{Hom}_k(\mathfrak{m}_{\mathbf{a}}/\mathfrak{m}_{\mathbf{a}}^2, k)$ is naturally the vector space dual to $\mathfrak{m}_{\mathbf{a}}/\mathfrak{m}_{\mathbf{a}}^2$, which is called the* cotangent space*.*

Proof. Consider the k-linear map sending the class of a polynomial to its linear term, regarded as a map restricted to the tangent space

$$\psi : I_X(\mathbf{a}) \to \mathrm{Hom}_k(T_{\mathbf{a}}X, k), \psi(f) = f_{1,\mathbf{a}}|_{T_{\mathbf{a}}X}.$$

By definition of the tangent space, this map is well-defined, that is, if $f \in I(X)$ then $\psi(f) = 0$. Moreover, note that ψ is surjective since any linear map on $T_{\mathbf{a}}X$ can be extended to a linear map on $\mathbb{A}^n_k$. So by the first isomorphism theorem it suffices to prove that $\mathrm{Ker}(\psi) = I(\mathbf{a})^2$.

Consider the vector subspace $W = \{g_{1,\mathbf{a}} \mid g \in I(X)\}$ of $k[x_1, \dots, x_n]_1$ and let d be its dimension. Then its zero locus $T_{\mathbf{a}}X$ has dimension $n - d$, and hence the space of linear forms vanishing on $T_{\mathbf{a}}X$ has dimension d again. As it clearly contains W, we conclude that W must be equal to the space of linear forms vanishing on $T_{\mathbf{a}}X$.

So if $f \in \mathrm{Ker}\,\psi$, i.e. the linear term of f vanishes on $T_{\mathbf{a}}X$,we know that there is a polynomial $g \in I(X)$ with $g_{1,\mathbf{a}} = f_{1,\mathbf{a}}$. But then $f - g$ has no constant or linear term, and hence we have in $k[X]$ that $\overline{f} = \overline{f - g} \in ((x_i - a_i)(x_j - a_j) \mid 1 \le i, j \le n) \subseteq I_X(\mathbf{a})^2$. This shows $\mathrm{Ker}\,\psi \subseteq I_X(\mathbf{a})^2$.

Conversely if $f, g \in I_X(\mathbf{a})$, then $(fg)_{1,\mathbf{a}} = f(\mathbf{a})g_{1,\mathbf{a}} + g(\mathbf{a})f_{1,\mathbf{a}} = 0 \cdot g_{1,\mathbf{a}} + 0 \cdot f_{1,\mathbf{a}} = 0$,and hence $\psi(fg) = 0$. This shows $I_X(\mathbf{a})^2 \subseteq \mathrm{Ker}\,\psi$.

Set $S = k[X] \setminus I_X(a)$. Then $\mathcal{O}_{X,\mathbf{a}} = S^{-1}k[X]$. We'll show

$$I_X(\mathbf{a})/I_X(\mathbf{a})^2 \cong S^{-1}(I_X(\mathbf{a})/I_X(\mathbf{a})^2) \cong S^{-1}I_X(\mathbf{a})/(S^{-1}I_X(\mathbf{a}))^2 = \mathfrak{m}_{\mathbf{a}}/\mathfrak{m}_{\mathbf{a}}^2.$$

The second isomorphism follows from flatness of localization. For the first isomorphism observe that the elements of S are already units in $I_X(\mathbf{a})/I_X(\mathbf{a})^2$ as follows: since $I_X(\mathbf{a})/I_X(\mathbf{a})^2$ is annihilated by $I(\mathbf{a})$, it is a $k[X]/I_X(\mathbf{a}) = k[X]/(x_1 - a_1, \ldots, x_n - a_n) = k$-module and since every element of S is a unit in $k[X]/I_X(\mathbf{a})$ it acts as a unit on $I_X(\mathbf{a})/I_X(\mathbf{a})^2$. In particular, for each $s \in S$ there exists an $s' \in S$ so that $ss' = 1$ in $k[X]/I_X(\mathbf{a})$ and thus $(1/s)I_X(\mathbf{a})/I_X(\mathbf{a})^2 = s'I(\mathbf{a})/I_X(\mathbf{a})^2 \subseteq I_X(\mathbf{a})/I_X(\mathbf{a})^2$. Thus localizing $I_X(\mathbf{a})/I_X(\mathbf{a})^2$ at S does not change anything. □

Remark 5.5. The previous Proposition implies that $\dim_k T_{\mathbf{a}}X = \dim_k \mathfrak{m}_{\mathbf{a}}/\mathfrak{m}_{\mathbf{a}}^2$. By Nakayama's Lemma the latter is the minimal number of generators of $\mathfrak{m}_{\mathbf{a}}$ also termed the *embedding dimension* of the ring $\mathcal{O}_{X,\mathbf{a}}$.

We can now make the tangent space into a functor.

Theorem 5.6. *There is a functor*

$$\langle\langle \textit{Affine varieties over } k \textit{ with a marked point} \rangle\rangle \to \langle\langle k\textit{-Vector spaces} \rangle\rangle$$

given on objects by $(X, \mathbf{a}) \mapsto T_{\mathbf{a}}X$ *and on regular maps* $h : X \to Y, h = (h_1, \ldots, h_m)$ *by the induced linear map*

$$dh : T_{\mathbf{a}}X \to T_{f(\mathbf{a})}Y, df(\mathbf{v}) = \left(\frac{\partial h_i}{\partial x_j}(\mathbf{a})\right)\mathbf{v}.$$

Proof. The proof is omitted. □

Monday, April 7 2025

5.1.2 Tangent cone

Definition 5.7. Let $X \subseteq \mathbb{A}_k^n$ be an affine variety and assume $\mathbf{a} \in X$. Consider the blow-up $\widetilde{X}$ of X at the ideal defining $\mathbf{a}$, $I(\mathbf{a}) = (x_1 - a_1, \ldots, x_n - a_n)$. Its exceptional set $\pi^{-1}(\mathbf{a})$ is then a projective variety. The affine cone of this exceptional set is called the *tangent cone* $C_{\mathbf{a}}X$ of X at $\mathbf{a}$. We will consider $C_{\mathbf{a}}X \subseteq \operatorname{Cone}(\mathbb{P}_k^{n-1}) = \mathbb{A}_k^n$ as a closed subvariety of the same ambient affine space as for X.

Remark 5.8. It turns out that if $0 \in X$ the tangent cone at 0 can be described as

$$C_0X = V^{\mathbb{A}}(f^{\text{in}} \mid f \in I(X)),$$

where f^{in} is the initial term of f or the homogeneous component of f of smallest degree. Unlike the tangent space it is not enough to take initial forms of generators for $I(X)$ to compute the tangent cone, however this suffices when $I(X)$ is principal.

Since $f_{1,0} = f^{\text{in}}$ when f has nonzero linear terms and otherwise $f_{1,0} = 0$ we see that $(f_{1,0} \mid f \in I(X)) \subseteq (f^{\text{in}} \mid f \in I(X))$ and so $C_0X \subseteq T_0X$. It is true more generally that $C_{\mathbf{a}}X \subseteq T_{\mathbf{a}}X$ for any $\mathbf{a} \in X$.

Example 5.9. The tangent cone of the parabola $X_1 = V^{\mathbb{A}}(y - x^2)$ at the origin is the horizontal axis $C_0X_1 = V^{\mathbb{A}}(y)$ while the tangent cone of $X_2 = V^{\mathbb{A}}(y^2 - x^2(x+1))$ is a union of two lines $C_0X_2 = V^{\mathbb{A}}(y^2 - x^2)$ and for $X_3 = V^{\mathbb{A}}(y^2 - x^3)$ we have $C_0X_3 = V^{\mathbb{A}}(y^2)$ is again the horizontal axis.

Note that unlike the tangent spaces, the tangent cones above have the same dimension as the respective varieties they are tangent to. We will see a formal reason for this below.

Theorem 5.10. *Let* $\mathbf{a}$ *be a point on an irreducible affine variety* X*. Then*

1. $\dim T_{\mathbf{a}}X \geq \dim X$ *and*

2. $\dim C_{\mathbf{a}}X = \dim X$.

More generally, if X *is not assumed irreducible:*

1. $\dim T_{\mathbf{a}}X \geq \dim_{\mathbf{a}} X$ *and*

2. $\dim C_{\mathbf{a}}X = \dim_{\mathbf{a}} X$,

where $\dim_{\mathbf{a}} X$ *is the local dimension of* X *at* $\mathbf{a}$*, i.e. the largest dimension of an irreducible component of* X *passing through* $\mathbf{a}$.

Proof. We may assume that $X \neq \{\mathbf{a}\}$, since otherwise $T_{\mathbf{a}}X = C_{\mathbf{a}}X = \{\mathbf{a}\} = X$ and the statement is trivial.

Let $d = \dim X$ and $n = \dim_k T_{\mathbf{a}}X = \dim_k \mathfrak{m}_{\mathbf{a}}/\mathfrak{m}_{\mathbf{a}}^2$, where $\mathfrak{m}_{\mathbf{a}}$ is the maximal ideal of $\mathcal{O}_{X,\mathbf{a}}$. By Nakayama's lemma, the ideal $\mathfrak{m}_{\mathbf{a}}$ is generated by n elements, say $\mathfrak{m}_{\mathbf{a}} = (f_1, \dots, f_n)$ with $f_i = \frac{g_i}{h_i} \in k[X]_{I_X(\mathbf{a})}$. Then there exists an affine neighborhood U of $\mathbf{a}$ in X such that $I_U(\mathbf{a}) = (f_1, \dots, f_n) =: I$. Indeed, if for some neighborhood W we have $I_W(\mathbf{a}) = (g_1, \dots, g_m)$ then $(g_1, \dots, g_m)\mathcal{O}_{X,\mathbf{a}} = (f_1, \dots f_n)$ means there exist $a_{ij}, b_{ij} \in \mathcal{O}_{X,\mathbf{a}}$ such that $g_i = \sum a_{ij} f_j$ and $f_i = \sum b_{ij} g_j$. Picking U so that $a_{ij}, b_{ij} \in k[U]$ (e.g. $U = D$(lcm of the denominators of a_{ij}, b_{ij})) yields the desired conclusion.

Let $\widetilde{X}_f$ be the blow-up of X at $f_1, \dots, f_n$ with canonical projection $\pi : \widetilde{X} \to X$. Since $\widetilde{X}_f$ is a closed subvariety of $X \times \mathbb{P}^{n-1}$, and so $\pi^{-1}(\mathbf{a}) \subseteq \{\mathbf{a}\} \times \mathbb{P}_k^{n-1}$, it has dimension $\dim \pi^{-1}(\mathbf{a}) \leq n - 1$ (as a projective variety). We will show however that $\dim \pi^{-1}(\mathbf{a}) = d - 1$.

We know from Proposition 4.4 that $\dim \widetilde{X} = \dim X = d$. The regular map $\pi : \widetilde{X} \to X$ has as its pullback the inclusion $\pi^* : R := k[X] \hookrightarrow k[\widetilde{X}] \cong \mathcal{R}(I)$. Then the ideal defining $\pi^{-1}(\mathbf{a})$ is $\pi^*(I) = I\mathcal{R}(I)$. We consider the open set $U_i = D(Y_i) \subseteq X \times \mathbb{P}^{n-1}$ and we want to determine the defining ideal of the affine variety $\pi^{-1}(\mathbf{a}) \cap U_i$ in $k[U_i] = R[\frac{Y_0}{Y_i}, \dots, \frac{Y_{i-1}}{Y_i}, \frac{Y_{i+1}}{Y_i}, \dots, \frac{Y_n}{Y_i}]$. Since on $\widetilde{X}$ we have $f_iY_j - f_jY_i = 0$ on U_i we have $\frac{Y_j}{Y_i} = \frac{f_j}{f_i}$. Thus we can rewrite

$$k[U_i] = R\left[\frac{f_0}{f_i}, \dots, \frac{f_{i-1}}{f_i}, \frac{f_{i+1}}{f_i}, \dots, \frac{f_n}{f_i}\right]$$

and we see that the ideal $Ik[U_i]$ is principal:

$$Ik[U_i] = (f_i)R\left[\frac{f_0}{f_i}, \dots, \frac{f_{i-1}}{f_i}, \frac{f_{i+1}}{f_i}, \dots, \frac{f_n}{f_i}\right]$$

as $f_j = f_i \cdot \frac{f_j}{f_i} \in (f_i)k[U_i]$ implies $Ik[U_i] \subseteq (f_i)k[U_i]$ and the opposite containment is evident. By Krull's principal ideal theorem we conclude that $\dim \pi^{-1}(\mathbf{a}) \cap U_i = \dim(U_i \cap \widetilde{X}_f) - 1 = d - 1$. Thus $\dim \pi^{-1}(\mathbf{a}) = d - 1 \leq n - 1$ implies $\dim X = d \leq \dim T_{\mathbf{a}} X = n$.

Since $C_{\mathbf{a}}X$ is the affine cone of $\pi^{-1}(\mathbf{a})$, we have $\dim C_{\mathbf{a}}X = \dim \pi^{-1}(\mathbf{a}) + 1 = d = \dim X$. □

Remark 5.11. By the localization argument in the proof of Theorem 5.4, the coordinate ring of the tangent cone can be identified with the *associated graded ring* of $R = k[X]$ with respect to $I = I_X(\mathbf{a})$ or of $\mathcal{O}_{X,\mathbf{a}}$ with respect to $\mathfrak{m}_{\mathbf{a}}$, respectively

$$\frac{\mathcal{R}(I)}{I\mathcal{R}(I)} = \frac{\bigoplus_{d\geq 0} I^i T^i}{\bigoplus_{d\geq 0} I^{i+1}T^i} = \bigoplus_{d\geq 0} I^i/I^{i+1}T^i = \operatorname{gr}_I(R) \cong \bigoplus_{d\geq 0} \mathfrak{m}_{\mathbf{a}}^i/\mathfrak{m}_{\mathbf{a}}^{i+1} = \operatorname{gr}_{\mathfrak{m}_{\mathbf{a}}} \mathcal{O}_{X,\mathbf{a}}.$$

Then part 2 of Theorem 5.10 can be restated as

$$\dim R = \dim \operatorname{gr}_I(R) \quad \text{and} \quad \dim X = \dim \operatorname{gr}_{\mathfrak{m}_{\mathbf{a}}} \mathcal{O}_{X,\mathbf{a}}.$$

5.2 Smoothness and the Jacobian criterion

Definition 5.12. A point $\mathbf{a}$ of a quasi-projective variety X is *smooth* or *nonsingular* if $C_{\mathbf{a}}X = T_{\mathbf{a}}X$. A quasi-projective variety X is said to be smooth or nonsingular if all points of X are nonsingular points of X.

Definition 5.13. A local ring R with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$ is called a *regular local ring* if its Krull dimension is equal to its embedding dimension, that is

$$\dim R = \dim_k \mathfrak{m}/\mathfrak{m}^2.$$

Theorem 5.14. *The following are equivalent for a point* $\mathbf{a}$ *of an affine variety* $X \subseteq \mathbb{A}^n_k$*:*

1. $\mathbf{a}$ *is a nonsingular point of* X
2. $\dim_k T_{\mathbf{a}}X = \dim_{\mathbf{a}} X$
3. $\mathcal{O}_{X,\mathbf{a}}$ *is a regular local ring*
4. *(Jacobian criterion) for* $I(X) = (f_1, \dots, f_s)$ *the rank of the* Jacobian matrix *is*

$$\operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(\mathbf{a})\right)_{1\leq i\leq s, 1\leq j\leq n} = n - \dim_{\mathbf{a}} X.$$

Proof. By definition, $\mathbf{a}$ is a nonsingular point of X iff $C_{\mathbf{a}}X = T_{\mathbf{a}}X$. Since $C_{\mathbf{a}}X \subseteq T_{\mathbf{a}}X$ and $T_{\mathbf{a}}X$ is irreducible, the two varieties are equal iff $\dim C_{\mathbf{a}}X = \dim T_{\mathbf{a}}X$ iff by Theorem 5.10 $\dim_{\mathbf{a}} X = \dim T_{\mathbf{a}}X$. This shows (1) $\iff$ (2).

From Theorem 5.4 we have $\dim T_{\mathbf{a}}X = \dim_k \mathfrak{m}_{\mathbf{a}}/\mathfrak{m}_{\mathbf{a}}^2$ thus (2) $\iff$ $\dim_k \mathfrak{m}_{\mathbf{a}}/\mathfrak{m}_{\mathbf{a}}^2 = \dim k[X]_{I(\mathbf{a})} = \dim \mathcal{O}_{X,\mathbf{a}}$.

From Lemma 5.2 we deduce rank $\left(\frac{\partial f_i}{\partial x_j}(\mathbf{a})\right) = n - \dim_k T_{\mathbf{a}}X$. Thus (2) $\iff$ (4). □

Example 5.15. For any $n, d \geq 0$ and any k, $\mathbb{A}_k^n$, $\mathbb{P}_k^n$ and $\mathrm{Gr}_k(n,d)$ are smooth. For $\mathbb{A}_k^n$ we can check this locally utilizing the Jacobian criterion. Since $\mathbb{P}_k^n$ and $\mathrm{Gr}_k(n,d)$ are locally isomorphic to affine space, it follows these varieties are smooth as well.

Wednesday, April 9, 2025

Remark 5.16. It is a result of commutative algebra that any regular local ring is an integral domain. Translating this into geometry using Theorem 5.14 this yields the statement that a variety is locally irreducible at every smooth point $\mathbf{a}$, i. e. that X has only one irreducible component meeting $\mathbf{a}$. Equivalently, any point on a variety at which two irreducible components meet is necessarily a singular point.

Remark 5.17. Because $\dim T_{\mathbf{a}}X \geq \dim_{\mathbf{a}} X$ holds, the rank of the Jacobiam matrix cannot exceed $n - \dim_{\mathbf{a}} X$, so the Jacobian criterion can be reformulated as follows:

> For X an algebraic variety with $I(X) = (f_1, \ldots, f_s)$ and $\mathbf{a} \in X$, $\mathbf{a}$ is smooth if and only if the rank of the *Jacobian matrix* is
>
> $$\operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(\mathbf{a})\right)_{1\leq i\leq s, 1\leq j\leq n} \geq n - \dim_{\mathbf{a}} X.$$

In the projective case we have a similar criterion.

Proposition 5.18 (The projective Jacobian criterion)**.** *Let $X \subseteq \mathbb{P}_k^n$ be a projective variety with $I(X) = (f_1, \ldots, f_s)$. A point $\mathbf{a} \in X$ is smooth if and only if*

$$\operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(\mathbf{a})\right) = n - \dim_{\mathbf{a}} X,$$

equivalently, if and only if rank $\left(\frac{\partial f_i}{\partial x_j}(\mathbf{a})\right) \geq n - \dim_{\mathbf{a}} X$.

In the above criterion, note that the entries of the Jacobian matrix are not well-defined because multiplying the coordinates of $\mathbf{a}$ by a scalar $\lambda \in k^*$ will multiply $\frac{\partial f_i}{\partial X_j}$ by $\lambda^{\deg(f_i)-1}$. However, these are just row transformations of the Jacobian matrix, which do not affect its rank. Hence the condition on rank in the projective Jacobi criterion is well-defined and can be checked on any representative of $[\mathbf{a}]$.

Definition 5.19. Let X be a variety. The *singular locus* of X, denoted X_{sing} is the set of non-smooth or singular points of X. The complement of the singular locus is the *smooth locus*.

The Jacobian criterion gives equations for the singular locus.

Example 5.20. Consider the projective quadric $Q = V^{\mathbb{P}}(X_0^2 + X_1^2 + \cdots + X_r^2)$ for some $0 \leq r \leq n$. In characteristic $\neq 2$ the Jacobian matrix at $\mathbf{a}$ is

$$\begin{bmatrix} 2a_0 & 2a_1 & \cdots & 2a_r & 0 \cdots & 0 \end{bmatrix}$$

and $\mathbf{a} \in Q_{\text{sing}}$ if and only if $a_0 = a_1 = \cdots = a_r = 0$. Thus

$$Q_{\text{sing}} = \{[0 : \cdots : 0 : a_{r+1} : \cdots : a_n] \mid [a_{r+1} : \cdots : a_n] \in \mathbb{P}_k^{n-r-1}\}$$

and Q is smooth if and only if $r = n$. In characteristic two, $X_0^2 + X_1^2 + \cdots + X_r^2 = (X_0 + \cdots + X_r)^2$ so we have $I(Q) = X_0 + \cdots + X_r$. The singular locus is the same as described above.

We now see that "most" points are smooth.

Theorem 5.21. *The set of smooth points of a quasi-projective variety is non-empty and Zariski open.*

Proof. Let $\mathbf{a} \in X$ be a singular point. By passing to a sufficiently small affine open set we may assume that X is affine and by Remark 5.16 that X is irreducible so that its local dimension is constant at all points, equal to $\dim X$. Then the Jacobian criterion gives that the singular locus of X is the vanishing set of the minors of size $n - \dim X$ of the Jacobian matrix of generators of $I(X)$. Thus the smooth locus is open.

We will prove that the smooth locus is non-empty here only in the case of a hypersurface $X = V(f) \subset \mathbb{A}^n$ for a non-constant irreducible polynomial $f \in k[x_1, \ldots, x_n]$. The general case can be reduced to this and any projective variety is birational to a hypersurface. Assume that all points of X are singular. Then the Jacobian matrix of f must have rank 0 at every point of X, which means that $\frac{\partial f}{\partial x_j}(\mathbf{a}) = 0$ for all $\mathbf{a} \in X$ and all $1 \leq j \leq n$. This yields that

$$\frac{\partial f}{\partial x_j} \in I(X) = (f) \qquad \forall 1 \leq j \leq n.$$

Since the degree of $\frac{\partial f}{\partial x_j}$ is smaller than that of f, it follows that $\frac{\partial f}{\partial x_j} = 0$ for all $1 \leq j \leq n$. In the case $\text{char} k = 0$ this is already a contradiction to f being non-constant. If $\text{char} k = p$ is positive, then f must be a polynomial in $x_1^p, \ldots, x_n^p$, and since the coefficients of f are also p-th powers (in an algebraically closed field), $f = g^p$ for some $g \in k[x_1, \ldots, x_n]$. This is a contradiction since f was assumed to be irreducible. □

Friday, April 11, 2025

5.3 Transverse intersection and Bertini's theorem

Lemma 5.22. *If X and Y are affine varieties and $\mathbf{a} \in X \cap Y$ then*

$$T_{\mathbf{a}}(X \cap Y) \subseteq T_{\mathbf{a}}(X) \cap T_{\mathbf{a}}(Y).$$

Proof. Recall that $I(X \cap Y) = \sqrt{I(X) + I(Y)}$, in particular $I(X \cap Y) \subseteq I(X) + I(Y)$. Moreover

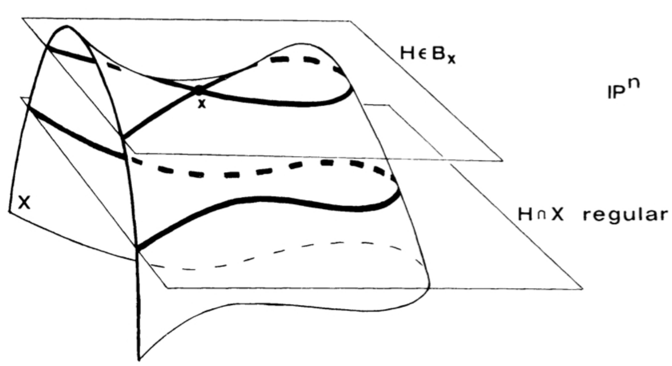
$$\begin{aligned} T_{\mathbf{a}}X &= V^{\mathbb{A}}(f_{1,\mathbf{a}} \mid f \in I(X)) \\ T_{\mathbf{a}}Y &= V^{\mathbb{A}}(f_{1,\mathbf{a}} \mid f \in I(Y)) \\ T_{\mathbf{a}}X \cap T_{\mathbf{a}}Y &= V^{\mathbb{A}}(f_{1,\mathbf{a}} \mid f \in I(X) \text{ or } f \in I(Y)) \\ &= V^{\mathbb{A}}(f_{1,\mathbf{a}} \mid f \in I(X) + I(Y)) \text{ by Lemma 5.2} \\ &\supseteq V^{\mathbb{A}}(f_{1,\mathbf{a}} \mid f \in I(X \cap Y)) = T_{\mathbf{a}}(X \cap Y). \quad \square \end{aligned}$$

Definition 5.23. Varieties X and Y are said to *intersect transversely* at a point $\mathbf{a} \in X \cap Y$ provided that $T_{\mathbf{a}}(X \cap Y) = T_{\mathbf{a}}(X) \cap T_{\mathbf{a}}(Y)$.

Example 5.24. The parabola $X = V^{\mathbb{A}}(y - x^2)$ and the line $Y = V^{\mathbb{A}}(y)$ do not intersect transversely at the origin since $T_0X = T_0Y = Y$, but $T_0(X \cap Y) = 0$. The parabola $X = V^{\mathbb{A}}(y - x^2)$ and any other line $Z = V^{\mathbb{A}}(y - c)$ with $c \in k^*$ intersect transversely.

The notion of transverse intersection is related to smoothness of intersections.

Example 5.25. Consider the "saddle" X pictured below. If we intersect it with a horizontal hyperplane H most of the time we obtain a smooth curve $X \cap H$. But if H goes through the low point of the saddle then the curve $X \cap H$ is singular at this point where it has a self-intersection.



Bertini's theorem formalizes the intuition in the preceding example.

Theorem 5.26 (Bertini's Theorem)**.** *Let $X \subseteq \mathbb{P}^n$ be a smooth projective variety. Then a general hyperplane $H \subseteq \mathbb{P}^n_k$ intersects X transversely and $X \cap H$ is smooth.*

Proof. Without loss of generality assume X is irreducible. Let $d = \dim X$. Consider the following incidence correspondence

$$\Sigma = \{(\mathbf{a}, H) \mid T_{\mathbf{a}}X \subseteq H\} \subseteq X \times \mathrm{Gr}(n, n+1) \cong X \times \mathbb{P}^n_k$$

Since X is smooth $T_{\mathbf{a}}X \cong \mathbb{A}^d_k$ and so its closure in projective space is $\overline{T_{\mathbf{a}}X} \cong \mathbb{P}^d_k$. Since H is Zariski closed and $T_{\mathbf{a}}X \subseteq H$ we have $\overline{T_{\mathbf{a}}X} \subseteq H$. Thus the fiber of $\pi_1 : \Sigma \to X$ at $\mathbf{a}$ is $\pi_1^{-1}(\mathbf{a}) \cong H/\overline{T_{\mathbf{a}}X} \cong \mathbb{P}^{n-1-d}$. By the theorem on dimension of fibers Σ is irreducible of dimension

$$\dim \Sigma = d + n - 1 - d = n - 1.$$

Since $\dim(\pi_2(\Sigma)) \leq \dim(\Sigma) = n - 1 < \dim \mathbb{P}^n_k$ we have that $\pi_2(\Sigma)$ is a proper closed subset of $\mathbb{P}^n_k$. Call its complement U. Then $U \neq \emptyset$ is Zariski open.

Claim 5.27. If $H \in U$ then $X \cap H$ is smooth and X intersects H transversely.

Lat $\mathbf{a} \in X \cap H$. We may replace X by an affine neighborhood of $\mathbf{a}$ and H by the intersection of H with this affine set, which is an affine hyperplane.

Since $T_{\mathbf{a}}X \not\subseteq H$ we have $T_{\mathbf{a}}(X \cap H) \subseteq T_{\mathbf{a}}X \cap T_{\mathbf{a}}H = T_{\mathbf{a}}X \cap H \subsetneq T_{\mathbf{a}}X$. So $\dim T_{\mathbf{a}}(X \cap H) < \dim T_{\mathbf{a}}X = d$.

On the other hand, by Theorem 5.10 $\dim T_{\mathbf{a}}(X \cap H) \geq \dim(X \cap H) = d - 1$. (The last equality follows by Krull's Height Theorem.) We conclude that $\dim T_{\mathbf{a}}(X \cap H) = \dim(X \cap H) = d - 1$ and thus by Theorem 5.14 that $X \cap H$ is smooth at $\mathbf{a}$.

As for the transverse intersection property, as $T_{\mathbf{a}}(X \cap H) \subseteq T_{\mathbf{a}}X \cap T_{\mathbf{a}}H = T_{\mathbf{a}}X \cap H \subsetneq T_{\mathbf{a}}X$ we have $d - 1 = \dim T_{\mathbf{a}}(X \cap H) \leq \dim T_{\mathbf{a}}X \cap T_{\mathbf{a}}H < T_{\mathbf{a}}X = d$ and so it follows that $\dim T_{\mathbf{a}}X \cap T_{\mathbf{a}}H = d - 1$ and consequently that $T_{\mathbf{a}}(X \cap H) = T_{\mathbf{a}}X \cap T_{\mathbf{a}}H$. $\square$

Chapter 6

Degree and Intersection

Monday, April 14 2025

A polynomial $f \in k[x]$ with coefficients in an algebraically closed field factors into linear factors as

$$f(x) = c(x - r_1)^{m_1}(x - r_2)^{m_2} \cdots (x - r_t)^{m_t}$$

where $r_1, \ldots, r_t \in k$ are the distinct roots of f and m_i is the multiplicity of the root r_i. Then $\deg(f) = m_1 + \cdots + m_t$ says that the degree of f is the number of roots it has, counted with multiplicity. Note that the roots of f can also be viewed as the intersection points of the graph of f, $V^{\mathbb{A}}(f(x)-y) \subseteq \mathbb{A}^2$ with the horizontal axis $V^{\mathbb{A}}(y)$.

In this chapter we define the degree of a projective variety $X \subseteq \mathbb{P}^n_k$. Classically, the degree is defined to be the number of intersection points of X with a general linear subvariety of $\mathbb{P}^n$ of dimension equal to the codimension of X. In a more algebraic approach, the degree is defined from the Hilbert polynomial of X. We will indicate why these two definitions are in fact equal and derive a classical bound on the degree of a nondegenerate variety (a variety which is not contained in a linear hyperplane of $\mathbb{P}^n_k$).

6.1 Hilbert function and Hilbert polynomial

Definition 6.1. An $\mathbb{N}$-*graded ring* is one that decomposes as a direct sum of abelian groups $R = \bigoplus_{i\geq 0} R_i$ so that $R_iR_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{N}$. In such a decomposition the elements of R_i are called the homogeneous elements of degree i of R.

An $\mathbb{N}$-*graded module* over the $\mathbb{N}$-graded ring R is a module that decomposes as a direct sum of abelian groups $M = \bigoplus_{i\geq 0} M_i$ so that $R_iM_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{N}$. In such a decomposition the elements of M_i are called the homogeneous elements of degree i of M.

Remark 6.2. If R is a graded ring and I is a homogeneous ideal of R then I is a graded R-module with $I_i = I \cap R_i$ and R/I is a graded R-module with $(R/I)_i = R_i/I_i$.

Example 6.3. In this section our main example of a graded ring will be the polynomial ring $R = k[X_0, \dots, X_n] = k^{\mathbb{P}}[\mathbb{P}^n_k]$ with R_i being the k-linear span of the monomials of degree i. Our main examples of graded modules will be quotient rings R/I with I a homogeneous ideal of R, e.g. $I = I(X)$ for a projective variety $X \subseteq \mathbb{P}^n_k$.

Definition 6.4. If R is an $\mathbb{N}$-graded ring with $R_0 = k$, a field, the *Hilbert function* of R is the function $h_R : \mathbb{N} \mapsto \mathbb{N} \cup \{\infty\}$ with values $h_R(i) := \dim_k R_i$.

Similarly, if M is an $\mathbb{N}$-graded R-module, we define the Hilbert function of M by $h_M : \mathbb{N} \mapsto \mathbb{N} \cup \{\infty\}, h_M(i) = \dim_k M_i$.

We define the *Hilbert series* of R or of an R-module M as above by $H_R(t) = \sum_{i\in\mathbb{N}} h_R(i)t^i$ and $H_M(t) = \sum_{i\in\mathbb{N}} h_M(i)t^i$.

Example 6.5. Consider the standard graded ring

$$R = k[x,y]/(x^2,y^3) = \underbrace{k}_{R_0} \bigoplus \underbrace{(kx \oplus ky)}_{R_1} \bigoplus \underbrace{(kxy \oplus ky^2)}_{R_2} \bigoplus \underbrace{kxy^2}_{R_3}.$$

Then $h_R(i) = \begin{cases} 1 & \text{if } t = 0 \\ 2 & \text{if } t = 1, 2 \\ 1 & \text{if } t = 3 \\ 0 & \text{if } t \geq 4 \end{cases}$ and $H_R(t) = 1 + 2t + 2t^2 + t^3$.

The key example of a Hilbert function is that of a polynomial ring.

Example 6.6. Let k be a field, and $R = k[X_0, \dots, X_n]$ be a polynomial ring with the standard grading: $\deg(X_i) = 1$ for each i. To compute the Hilbert function, we need to compute the size of a k-basis for R_i for each i. We have

$$R_i = \bigoplus_{a_0+\cdots+a_n=i} kX_0^{a_0}\cdots X_n^{a_n}.$$

We can find a bijection between these monomials and the set of strings that contain i stars and n bars, where the monomial $X_0^{a_0}\cdots X_n^{a_n}$ corresponds to the string with a_0 stars, then a bar, then a_1 stars, a bar, etc. Thus, the number of monomials is the number of ways to choose n bars from $i+n$ spots, i.e.,

$$h_R(i) = \binom{i+n}{n} \quad \text{for } i \geq 0.$$

We observe the binomial function here can be expressed as a polynomial in i for $i \geq 0$; let

$$P_n(i) = \frac{(i+n)(i+n-1)\cdots(i+1)}{n!} \in \mathbb{Q}[i].$$

Observe that $P_n(i)$ has $-1, \dots, -n$ as roots. Then we have

$$h_R(i) = \begin{cases} P_n(i) & \text{if } i \geq -n \\ 0 & \text{if } i < 0. \end{cases}$$

Note that the two cases overlap for $t = -n, \dots, -1$.

Notice that in this example **the Hilbert function is eventually (for $i \geq -n$) equal to a polynomial of degree n.** This polynomial is called the Hilbert polynomial, so the Hilbert polynomial of R is P_n. Moreover note that $R = k[\mathbb{P}^n_k]$ and $\dim \mathbb{P}^n_k = n = \deg(P_n)$.

To compute the Hilbert series, notice that the number of monomials of degree i is equal to the number of ordered tuples $(a_1, \dots, a_n)$ with $\sum_{j=0}^n a_j = i$. This is the coefficient of t^i in the product

$$(1 + t + t^2 + \cdots + t^{a_0} + \cdots)(1 + t + t^2 + \cdots + t^{a_1} + \cdots) \cdots (1 + t + t^2 + \cdots + t^{a_n} + \cdots)$$

hence

$$H_R(t) = (1 + t + t^2 + \cdots + t^i + \cdots)^n = \frac{1}{(1-t)^{n+1}}.$$

Notice that the power of $(1-t)$ in the denominator is $n + 1 = \dim R$.

A very important property of vector space dimension that makes the theory of Hilbert functions work is its additivity on short exact sequences:

Lemma 6.7. *If L, M, N are graded R-modules that form a short exact sequence $0 \to L \to M \to N \to 0$ and if the maps in this sequence preserve degrees, then there are equalities $h_M(i) = h_L(i) + h_N(i)$ for all $i \in \mathbb{N}$ and*

$$H_M(t) = H_L(t) + H_N(t).$$

Proof. Because the maps are assumed degree-preserving, the sequence $0 \to L \to M \to N \to 0$ restricts for each $i \in \mathbb{N}$ to an exact sequence of graded pieces $0 \to L_i \to M_i \to N_i \to 0$. Since $L_i = \operatorname{Ker}(M_i \to N_i)$ by exactness and since the rank of the vectors space map $M_i \to N_i$ is equal to $\dim_k N_i$ by surjectivity, the rank-nullity formula gives

$$\dim_k M_i = \dim_k \operatorname{Ker}(M_i \to N_i) + \operatorname{rank}(M_i \to N_i) = \dim_k L_i + \dim_k N_i.$$

Multiplying the above identity by t^i and summing over all $i \in \mathbb{N}$ yields $H_M(t) = H_L(t) + H_N(t)$. □

We can generalize Example 6.6 as follows.

Theorem 6.8 (Hilbert-Serre)**.** *Let $R = k[X_0, \dots, X_n]$. The Hilbert series $H_M(t)$ of any finitely generated R-module M is a rational function of the form*

$$H_M(t) = \frac{f(t)}{(1-t)^{n+1}} \text{ with } f \in \mathbb{Z}[t].$$

Wednesday, April 16 2025

Proof. By induction on n.

If $n = -1$ then $R = k$ and M is a finitely generated k-vector space, so that $\dim_k M_i = 0$ for $i \gg 0$ (specifically for $i >$ the maximum degree of a generator of M). Thus $H_M(t) = f(t)$ for some $f \in \mathbb{N}[t]$. (In this case the Hilbert series is a polynomial.)

For the induction step, multiplication by x_n on M induces an exact sequence

$$0 \to K \to M \xrightarrow{x_n} M \to N \to 0$$

where $K = \{m \in M \mid x_n m = 0\}$ and $N = M/(x_n)M$. Because K and N are annihilated by x_n they are finitely generated modules over $R/(x_n) = R_0[x_1, \ldots, x_{n-1}]$. The exact sequence above decomposes into a direct sum of exact sequences of vector spaces of the form

$$0 \to K_i \to M_i \xrightarrow{x_n} M_{i+1} \to N_{i+1} \to 0$$

Additivity of length yields

$$\dim_k(K_i) - \dim_k(M_i) + \dim_k(M_{i+1}) - \dim_k(N_{i+1}) = 0$$

or

$$h_K(i) - h_M(i) + h_M(i+1) - h_N(i+1) = 0.$$

In terms of Hilbert series this gives after multiplying by t^{i+1} and summing up

$$tH_K(t) - tH_M(t) + H_M(t) - H_N(t) = 0$$

or

$$(1-t)H_M(t) = H_N(t) - tH_K(t).$$

Applying the inductive hypothesis for $H_N(t)$ and $H_K(t)$ and substituting into the above identity yields the desired conclusion. □

Remark 6.9. The expression for $H_M(t)$ in the Hilbert-Serre theorem may not be in reduced form. The reduced form will be

$$H_M(t) = \frac{f(t)}{(1-t)^d} \text{ with } f \in \mathbb{Z}[t], f(1) \neq 0 \text{ and } d \geq 0. \quad (6.1)$$

It is a fact best proven in a commutative algebra course that in the reduced form above $d = \dim(M)$ is the Krull dimension of M.

Corollary 6.10. *Let R be a standard graded polynomial ring. The Hilbert function $h_M(i)$ of any finitely generated R-module M is given for sufficiently large i by a polynomial $P_M(i) \in \mathbb{Q}[i]$ of degree* $\dim(M) - 1$ *having leading coefficient $f(1)/(dim(M)-1)!$ for f as in (6.1).*

Proof. Use the formula for the reduced Hilbert series

$$H_M(t) = \frac{f(t)}{(1-t)^d} \text{ with } h \in \mathbb{Z}[t], f(1) \neq 0 \text{ and } d = \dim(M).$$

Substituting in the formula above and the "negative binomial" formula of Example 6.6

$$\frac{1}{(1-t)^d} = \sum_{i=0}^{\infty} \binom{i+d-1}{d-1} t^i,$$

where

$$P_{d-1}(i) = \binom{i+d-1}{d-1}$$

is a polynomial of degree $d-1$ with leading coefficient $\frac{1}{(d-1)!}$, and $f(t) = \sum_{j=0}^{s} c_j t^j$ we get

$$H_M(t) = \left(\sum_{j=0}^{s} c_j t^j\right)\left(\sum_{i=0}^{\infty} P_{d-1}(i) t^i\right).$$

For $i \geq \deg(f)$ we see by identifying the coefficient of t^i on both sides of the equation above that

$$h_M(i) = \sum_{j=0}^{s} c_j P_{d-1}(i-j) =: P_M(i)$$

is a polynomial in i of degree $d-1$ with leading coefficient $\frac{\sum_{j=0}^{s} c_j}{(d-1)!} = \frac{f(1)}{(d-1)!} \neq 0$ since $f(1) \neq 0$. □

Definition 6.11. The *Hilbert polynomial* of a finitely generated module over a standard graded polynomial ring is the polynomial $P_M(i) \in \mathbb{Q}[i]$ that agrees with $h_M(i)$ for $i \gg 0$.

Exercise 6.12. Suppose that I, J are homogeneous ideals in the same polynomial ring such that $I^{\text{sat}} = J^{\text{sat}}$. Then $P_{S/I} = P_{S/J}$.

Definition 6.13. The *multiplicity* of a finitely generated module M with $\dim(M) = d$ over a standard graded polynomial ring is $e(R) = (d-1)!$ times the leading coefficient of the Hilbert polynomial $P_M(i)$.

From Corollary 6.10 we get a quick way of computing multiplicity

Corollary 6.14. *The multiplicity of a finitely generated module M with $\dim(M) = d$ and Hilbert series in reduced form*

$$H_M(t) = \frac{f(t)}{(1-t)^d} \textit{ with } f \in \mathbb{Z}[t], f(1) \neq 0 \textit{ and } d \geq 0.$$

is $e(M) = f(1)$.

Example 6.15. The Hilbert polynomial of a standard graded ring $R = k[X_0, \dots, X_n]$ is $P_n(i) = \frac{(i+n)(i+n-1)\cdots(i+1)}{n!} \in \mathbb{Q}[i]$ as discussed in Example 6.6. The multiplicity of R is $e(R) = n! \cdot \frac{1}{n!} = 1$.

Example 6.16. Let $f \in R = k[X_0, \dots, X_n]$ be a homogeneous polynomial of degree d. From the short exact sequence $0 \to R \xrightarrow{\cdot f} R \to R/(f) \to 0$ we obtain

$$H_{R/(f)}(t) = H_R(t) - t^d H_R(t) = \frac{(1-t^d)}{(1-t)^{n+1}} = \frac{(1 + t + \cdots + t^{d-1})}{(1-t)^n}.$$

Then $e(R/(f)) = d$.

Exercise 6.17. More generally, if R is a graded ring and $f_1, \dots, f_r$ is a homogeneous regular sequence with $\deg(f_i) = d_i$ then $e(R/(f_1, \dots, f_r)) = d_1 \cdots d_r \cdot e(R)$.

6.2 The degree of a projective variety

Definition 6.18. Suppose that $X \subseteq \mathbb{P}^n_k$ is a projective variety. We define the *Hilbert polynomial* P_X of X to be the Hilbert polynomial $P_{k[X_0,\dots,X_n]/I^{\mathbb{P}}(X)}$. By Corollary 6.10, this polynomial has degree $d = \dim X$. We define the *degree* $\deg(X)$ of X to be $d!$ times the leading coefficient of P_X.

Example 6.19. By Example 6.15, $\deg(P^n_k) = 1$.

Example 6.20. Let f be a homogeneous polynomial of degree d. By Example 6.16, the hypersurface $V^{\mathbb{P}}(f)$ has degree $\deg(V^{\mathbb{P}}(f)) = d$.

Friday, April 18, 2025

Remark 6.21. The notion of degree is not an isomorphism invariant of a projective variety, but rather depends on the ambient projective space that the variety is embedded in. For example, we know that the Veronese variety is isomorphic to $\mathbb{P}^n_k$, but the degree of the Veronese variety is not equal to 1 (exercise).

Example 6.22. The Veronese surface (see Examples 2.62) $V_{2,2} \subseteq \mathbb{P}^5_k$ is isomorphic to $\mathbb{P}^2_k$ so it has dimension 2 and codimension $5 - 2 = 3$. The coordinate ring of $V_{2,2}$ is $k[x^2, xy, xz, y^2, yz, z^2]$ with the degree of each element being half of what it is in $k[x, y, z]$. Then $P_{V_{2,2}}(i) = P_{\mathbb{P}^2}(2i) = \binom{2i+2}{2} = (i+1)(2i+1) = 2i^2 + 3i + 1$. Since the leading coefficient of the Hilbert Polynomial is 2, we have $\deg V_{2,2} = 2! \cdot 2 = 4$.

Exercise 6.23. The following are true:

1. The degree of a nonempty variety is a positive integer.
2. Suppose that $Y = Y_1 \cup Y_2$ is the union of two projective varieties of $\dim Y_1 = \dim Y_2 = d$ such that $\dim(Y_1 \cap Y_2) < d$. Then $\deg(Y) = \deg(Y_1) + \deg(Y_2)$.

3. If $X = \bigcup_{i=1}^r X_i$ is the decomposition of a variety X with $\dim X = d$ into distinct irreducible components then $\deg(X) = \sum_{\dim(X_i)=d} \deg(X_i)$.
4. The degree of a point in $\mathbb{P}_k^n$ is 1.
5. The degree of a finite set of points $X \subseteq \mathbb{P}_k^n$ is $\deg(X) = \#X$.

We now see that the degree of a projective variety can be computed by intersection with a general linear subspace of complementary dimension. The proof relies on

Lemma 6.24. *If $V_1, \dots V_r$ are proper subspaces of a vector space V then $\bigcup_{i=1}^r V_i \neq V$.*

and the following improved version of Krull's height theorem

Lemma 6.25. *Suppose that R is a Noetherian commutative ring and $\ell \in R$ is not a unit. Then $\dim R \leq \dim R/(\ell)R + 1$. If ℓ is not contained in any of the minimal primes of R then $\dim R = \dim R/(\ell)R + 1$.*

Theorem 6.26. *Suppose that $\emptyset \neq X \subseteq \mathbb{P}_k^n$ is a projective variety such that all irreducible components of X have dimension d. Then*

1. *for a general hyperplane $H \subseteq \mathbb{P}_k^n$, we have that*
   - *(a) H does not contain any irreducible component of X,*
   - *(b) $X \cap H$ is equidimensional (all irreducible components have the same dimension) with $\dim(X \cap H) = \dim(X) - 1$*
   - *(c) X and H intersect transversely at a general point of every irreducible component of $X \cap H$*
   - *(d) $\deg(X) = e(k[X_0, \dots, X_n]/(I(X) + I(H)))$*
2. *If L is a general linear subvariety of $\mathbb{P}_k^n$ of codimension d ($\dim L = n - d$) then*
   - *(a) $X \cap L$ consists of finitely many points*
   - *(b) X intersects L transversely at each intersection point.*

   If L is a linear space that satisfies (a) and (b) above then $\deg(X) = \#(X \cap L)$.

Proof. 1. (a) Consider $S = k[X_0, \cdots, X_n]$, the homogeneous coordinate ring of X $k^{\mathbb{P}}[X] = S/I^{\mathbb{P}}(X)$ and the prime decomposition of the (radical) ideal defining X $I^{\mathbb{P}}(X) = P_1 \cap \cdots \cap P_r$, where P_i are homogeneous prime ideals in S. Since $X \neq \emptyset$ each $P_i \subsetneq (X_0, \dots, X_n)$.

Claim 6.27. There is a linear form $\ell \in k^{\mathbb{P}}[X]_1$ such that $\ell \neq 0$ and ℓ is not a zero divisor.

The statements $\ell \neq 0$ and ℓ is not a zero divisor are equivalent to $\ell \notin \bigcup_{i=1}^r [P_i]_1$. (This is an open set of S_1). Since each $[P_i]_1$ is a proper vector subspace of S_1, Lemma 6.24 applies and gives the existence of $\ell \in S_1 \setminus \bigcup_{i=1}^r [P_i]_1$, as desired. Since $\ell \notin P_i$ we have that $V(\ell)$ does not contain $V(P_i)$.

(b) Set $H = V^{\mathbb{P}}(\ell)$. By Lemma 6.25 we have $\dim k[X \cap H] = \dim k[X]/(\ell)k[X] = \dim k[X]-1$, thus all minimal primes of the ideal generated by ℓ in $k[X]$ have height one.

Thus $X \cap H = V_X(\ell)$ is equidimensional of codimension one, that is, of $\dim(X \cap H) = d - 1$.

(c) This follows by an argument similar to that in the proof of Bertini's Theorem 5.26. Here by "a general point of an irreducible component of $X \cap H$" we mean a smooth point of X that belongs to that component.

(d) We now show that $\deg(X) = e(k[X_0, \ldots, X_n]/(I(X) + I(H)))$ using the short exact sequence

$$0 \to k[X] \xrightarrow{\cdot \ell} k[X] \to k[X]/(\ell) \to 0.$$

It follows from here that $H_{k[X]}(t) = tH_{k[X]}(t) + H_{k[X]/(\ell)}(t)$ so if $H_{k[X]}(t) = \frac{f(t)}{(1-t)^{d+1}}$ in reduced form then

$$H_{k[X]/(\ell)}(t) = (1-t)H_{k[X]}(t) = \frac{f(t)}{(1-t)^d}$$

in reduced form so that $\deg(X) = f(1) = e(k[X]/(\ell))$. Finally, we have that $k[X]/(\ell) = k[X_0, \ldots, X_n]/(I(X) + I(H))$.

2. Part 2 follows from 1 by induction on d. Repeating the reasoning in part 1 d times we obtain a sequence of linear forms $\ell_1, \ldots, \ell_d$ so that for $L = V^{\mathbb{P}}(\ell_1, \ldots, \ell_d)$ we have

- $\dim(X \cap L) = 0$ so that $X \cap L$ is a finite set of points
- X intersects L transversely at each intersection point
- $\deg(X) = e(k[X_0, \ldots, X_n]/(I(X) + I(L))$.

As a last step we will show $I(X \cap L) = (I(X) + I(L))^{\mathrm{sat}}$. This holds if and only if for each $\mathbf{a} \in X \cap L$ we have $(I(X) + I(L))_{I(\mathbf{a})} = I(\mathbf{a})_{I(\mathbf{a})} = \mathfrak{m}_{\mathbf{a}}$ in $\mathcal{O}_{\mathbb{P}^n_k}(\mathbf{a})$.

Since X intersects L transversely at $\mathbf{a}$ it means that $T_{\mathbf{a}}X \cap T_{\mathbf{a}}L = T_{\mathbf{a}}\mathbf{a} = \{\mathbf{a}\}$ is a 0-dimensional vector space. This means that

$$V(f_{1,\mathbf{a}} \mid f \in I(X)) \cap V(f_{1,\mathbf{a}} \mid f \in I(L)) = \{\mathbf{a}\}$$

so $V(f_{1,\mathbf{a}} \mid f \in I(X) + I(L)) = \{\mathbf{a}\}$ and thus $(f_{1,\mathbf{a}} \mid f \in I(X) + I(L)) = I(\mathbf{a})$ (by linear algebra because both ideals are generated by linear forms). Then

$$(I(X) + I(L))_{I(\mathbf{a})}/\mathfrak{m}_{\mathbf{a}}^2 = \mathfrak{m}_{\mathbf{a}}/\mathfrak{m}_{\mathbf{a}}^2$$

yields by Nakayama that $(I(X) + I(L))_{I(\mathbf{a})}$ contains a set of minimal generators for $\mathfrak{m}_{\mathbf{a}}$ so the desired conclusion $(I(X) + I(L))_{I(\mathbf{a})} = \mathfrak{m}_{\mathbf{a}}$ follows.

Now by Exercises 6.12 and 6.23 (5) we have

$$\begin{aligned} \#(X \cap L) &= \deg(I \cap X) = e(k[X_0, \ldots, X_n]/I(X \cap L)) \\ &= e(k[X_0, \ldots, X_n]/(I(X) + I(L)) = \deg(X \cap L). \end{aligned}$$

6.2.1 Multiplicity bound

We now give a bound for the multiplicity of a non-degenerate projective variety.

Theorem 6.28. *Suppose that X is a projective subvariety of $\mathbb{P}^n_k$ and X is not contained in a linear hyperplane (X is nondegenerate). Then*

$$\deg(X) \geq \operatorname{codim}(X) + 1.$$

Proof. The proof is omitted. □

Definition 6.29. A nondegenerate projective variety X satisfying $\deg(X) = \operatorname{codim}(X)+1$ is called a *variety of minimal multiplicity*.

Varieties of minimal multiplicity have been classified by Del Pezzo and Bertini. The smooth ones are projectively equivalent to one of the following

- a quadric hypersurface $V^{\mathbb{P}}(Q)$ with Q a homogeneous quadratic polynomial,
- the Veronese surface $V_{2,2} \subseteq \mathbb{P}^5_k$ (see Examples 2.62 and 6.22),
- a rational normal scroll.

Moreover a singular variety of minimal degree is a cone over one of the above (same defining ideal in a larger polynomial ring).

□

Monday, April 21 2025

6.3 Bezout's theorem

Definition 6.30. The *ruled join* of two projective varieties $X \subseteq \mathbb{P}^n_k$ and $Y \subseteq \mathbb{P}^m_k$ is obtained as follows: embed $\mathbb{P}^n_k$ and $\mathbb{P}^m_k$ into $\mathbb{P}^{m+n+1}$ using the maps

$$\begin{aligned} \iota_1 : \mathbb{P}^n_k \to \mathbb{P}^{m+n+1} \qquad & \iota_1([a_0 : \cdots : a_n]) = [a_0 : \cdots : a_n : 0 : \cdots : 0] \\ \iota_2 : \mathbb{P}^m_k \to \mathbb{P}^{m+n+1} \qquad & \iota_2([b_0 : \cdots : b_m]) = [0 : \cdots : 0 : b_0 : \cdots : b_m] \end{aligned}$$

and define the ruled join $J(X,Y)$ to be the union of all lines in $\mathbb{P}^{m+n+1}$ connecting a point on $\iota_1(X)$ with a point on $\iota_2(Y)$.

The line connecting $[\mathbf{u}]$ and $[\mathbf{v}]$ is the set of points $\{[s\mathbf{u} + t\mathbf{v}] \mid [s : t] \in \mathbb{P}^1\}$.

We will use the following

Lemma 6.31. *If A, B are graded k-algebras then $e(A \otimes_k B) = e(A) \cdot e(B)$.*

Proof. As a graded k-algebra the tensor product decomposes as follows

$$A \otimes_k B := \bigoplus_{\ell \geq 0} \left[\bigoplus_{i+j=\ell} A_i \otimes_k B_j \right]$$

whence

$$h_{A \otimes_k B}(\ell) = \sum_{i+j=\ell} \dim_k A_i \otimes_k B_j = \sum_{i+j=\ell} h_A(i) h_B(j).$$

This gives rise to the identity of Hilbert series

$$H_{A\otimes_k B}(t) = H_A(t) \cdot H_B(t).$$

Writing the right hand side in reduced form according to Remark 6.9 gives

$$H_{A\otimes_k B}(t) = \frac{f_A(t)}{(1-t)^{\dim A}} \cdot \frac{f_B(t)}{(1-t)^{\dim B}} = \frac{f_A(t) f_B(t)}{(1-t)^{\dim A\otimes_k B}}.$$

Here we have used the fact that $\dim(A \otimes_k B) = \dim A + \dim B$. Now recall that multiplicity is obtained by evaluating the numerator of the Hilbert series in reduced form. So we conclude

$$e(A \otimes_k B) = f_A(1) f_B(1) = e(A) \cdot e(B).$$

□

Proposition 6.32. *If $X \subseteq \mathbb{P}^n_k$ and $Y \subseteq \mathbb{P}^m_k$ are projective varieties with defining ideals $I(X)$ in $k[X_0, \dots, X_n]$ and $I(Y)$ in $k[Y_0, \dots, Y_m]$ and $S = k[X_0, \dots, X_n, Y_0, \dots, Y_m]$, their join satisfies:*

1. $J(X,Y) = \iota_1(X) \cup \iota_2(Y) \cup \{[\mathbf{a}, \mathbf{b}] \mid [\mathbf{a}] \in X, [\mathbf{b}] \in Y\}$
2. $J(X,Y) = V^{\mathbb{P}}(I(X)S + I(Y)S)$ *is a projective variety*
3. $\dim J(X,Y) = \dim(X) + \dim(Y) + 1$
4. $\deg J(X,Y) = \deg X \cdot \deg Y$
5. *for* $[\mathbf{a}] \in X, [\mathbf{b}] \in Y$, *we have* $T_{[\mathbf{a},\mathbf{b}]} J(X,Y) = \{[\mathbf{u}, \mathbf{v}] \mid \mathbf{u} \in T_{[\mathbf{a}]} X, \mathbf{v} \in T_{[\mathbf{b}]} Y\}$.

Proof. 1. This follows as the line connecting $[\mathbf{a}, 0] \in \iota_1(X)$ for some $[\mathbf{a}] \in X$ to $[0, \mathbf{b}] \in \iota_2(Y)$ for some $[\mathbf{b}] \in Y$ to can be written as

$$\begin{aligned}\{[s\iota_1(\mathbf{a}) + t\iota_2(\mathbf{b})] \mid [s:t] \in \mathbb{P}^1\} &= \{[\mathbf{a}, 0]\} \cup \{[0, \mathbf{b}]\} \cup \{[s\mathbf{a}, t\mathbf{b}] \mid s, t \in k^*\} \\ &= \{\iota_1([\mathbf{a}])\} \cup \{\iota_2([\mathbf{b}])\} \cup \{[s\mathbf{a}, t\mathbf{b}] \mid [s\mathbf{a}] = [\mathbf{a}] \in X, [t\mathbf{b}] = [\mathbf{b}] \in Y\}.\end{aligned}$$

2. This follows since

$$V^{\mathbb{P}}(I(X)S + I(Y)S) \quad = \quad V^{\mathbb{P}}(I(X)S) \cap V^{\mathbb{P}}(I(Y)S)$$

consists of pairs $[\mathbf{a}, \mathbf{b}]$ where $[\mathbf{a}] \in X$ or $\mathbf{a} = 0$ and $[\mathbf{b}] \in Y$ or $\mathbf{b} = 0$, which is the same as the description of $J(X,Y)$ in part 1. (Note that $\mathbf{a}$ and $\mathbf{b}$ can't be simultaneously 0.)

3. We know that $C(J(X,Y)) = V^{\mathbb{A}}(I(X)S + I(Y)S)$ and by that

$$V^{\mathbb{A}}(I(X)S + I(Y)S) = V^{\mathbb{A}}(I(X)) \times V^{\mathbb{A}}(I(Y)) = C(X) \times C(Y).$$

Thus , $\dim C(J(X,Y)) = \dim C(X) \times C(Y) = (\dim(X)+1) + (\dim(Y)+1) = \dim(X) + \dim(Y) + 2$. Then $\dim J(X,Y) = \dim C(J(X,Y)) - 1 = \dim(X) + \dim(Y) + 1$.

4. We have shown that $C(J(X,Y)) = C(X) \times C(Y)$, thus $k[C(J(X,Y))] \cong k[C(X)] \otimes_k k[C(Y)]$. As the coordinate ring of a projective variety and its cone are the same, this yields $k[J(X,Y)] \cong k[X] \otimes_k k[Y]$.

By Lemma 6.31 we see that

$$\deg J(X,Y) = e(k[J(X,Y)]) = e(k[X] \otimes_k k[Y]) = e(k[X]) \times e(k[Y]) = \deg(X) \cdot \deg(Y).$$

5. For this part note that $I(X)S + I(Y)S = I(J(X,Y))$ since $I(X)$ and $I(Y)$ are written in disjoint sets of variables. This yields that

$$T_{[\mathbf{a},\mathbf{b}]} J(X,Y) = V(f_{1,\mathbf{a}}, g_{1,\mathbf{b}} \mid f \in I(X), g \in I(Y)) = \{[\mathbf{u},\mathbf{v}] \mid \mathbf{u} \in T_{[\mathbf{a}]}X, \mathbf{v} \in T_{[\mathbf{b}]}Y\}.$$

□

Wednesday, April 23 2025

We are now ready to prove Bezout's Theorem.

Theorem 6.33 (Bezout's Theorem - weak form)**.** *If X and Y are projective varieties in $\mathbb{P}^n_k$ that intersect transversely in finitely many points, then $\#(X \cap Y) = \deg X \cdot \deg Y$.*

Proof. Consider the ruled join $J(X,Y) \subseteq \mathbb{P}^{2n+1}_k$ as introduced in Definition 6.30. Consider the linear space $L = V^{\mathbb{P}}(X_0 - Y_0, \ldots, X_n - Y_n)$. Then we have

$$J(X,Y) \cap L = \{[\mathbf{a},\mathbf{a}] \mid [\mathbf{a}] \in X \cap Y\}.$$

Since the sets $X \cap Y$ are $J(X,Y) \cap L$ and clearly in bijection, we have $\#(X \cap Y) = \#(J(X,Y) \cap L)$.

We claim that L and $J(X,Y)$ intersect transversely. Indeed, we know by Proposition 6.32 (5) that at $[\mathbf{a},\mathbf{a}] \in J(X,Y) \cap L$, equivalently $[\mathbf{a}] \in X \cap Y$, the tangent space is given by

$$T_{[\mathbf{a},\mathbf{a}]} J(X,Y) = \{[\mathbf{u},\mathbf{v}] \mid \mathbf{u} \in T_{[\mathbf{a}]}X, \mathbf{v} \in T_{[\mathbf{a}]}Y\}.$$

and the tangent space to L is L. Consequently

$$T_{[\mathbf{a},\mathbf{a}]} J(X,Y) \cap T_{[\mathbf{a},\mathbf{a}]} L = \{[\mathbf{u},\mathbf{u}] \mid \mathbf{u} \in T_{[\mathbf{a}]}X \cap T_{[\mathbf{a}]}Y\}$$

Since X and Y intersect transversely at $[\mathbf{a}]$ we have $T_{[\mathbf{a}]}X \cap T_{[\mathbf{a}]}Y = T_{[\mathbf{a}]}\{[\mathbf{a}]\} = \{[\mathbf{a}]\}$, which yields that $T_{[\mathbf{a},\mathbf{a}]} J(X,Y) \cap T_{[\mathbf{a},\mathbf{a}]} L = \{[\mathbf{a},\mathbf{a}]\}$. We have thus proven the claim regarding transverse intersection.

Now we see that L satisfies conditions (a) and (b) in part 2 of Theorem 6.26. By the last sentence of that theorem we conclude

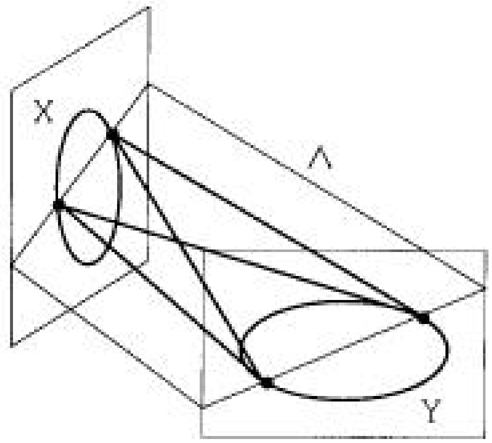
$$\deg J(X,Y) = \#(J(X,Y) \cap L).$$

Putting this together with Proposition 6.32 (4) and the description of $J(X,Y) \cap L$ we have

$$\deg X \cdot \deg Y = \deg J(X,Y) = \#(J(X,Y) \cap L) = \#(X \cap Y).$$

□

Below is an illustration of the proof (with L denoted Λ).



Friday, April 25 2025

Next we want to prove a version of Bezout's theorem "with mutiplicities". To do so, we want to come up with a reasonable way to count the multiplicity of an intersection point of two varieties. Here we will take this to be the length of a certain local ring at that point. Be warned that this is not the best definition for intersection multiplicity, but it is the best that we can do in this class. A better definition (Serre multiplicity) requires homological algebra.

Recall the following definitions from Math 905:

Definition 6.34. Let R be a ring and M a R-module.

- M is *simple* if it is nonzero and M has no nontrivial proper submodules.
- A *composition series* for M of length n is a chain of submodules

$$M = M_n \supsetneq M_{n-1} \supsetneq M_{n-2} \cdots M_1 \supsetneq M_0 = 0$$

  with M_i/M_{i-1} simple for all $i = 1, \dots, n$.
- M has *finite length* if it admits a finite composition series.
- The length of M, denoted $\ell_R(M)$ is the minimal length n of a composition series for M.

Example 6.35. If R is a local ring with maximal ideal $\mathfrak{m}$ then $k = R/\mathfrak{m}$ is the only simple module. Am R-module M has finite length if and only if $\operatorname{Ass}(M) = \{\mathfrak{m}\}$.

In this case M has length r if and only if it has a simple filtration whose quotients are r copies of k of and only if $M \cong k^r$ as a k-vector space (but not necessarily as R-modules), that is, $\ell_R M = \dim_k M$.

Recall also from Math 905 the notion of prime filtration and its relationship to length.

Definition 6.36. Let R be a Noetherian ring and M a finitely generated R-module. Then there exists a finite chain of submodules

$$M = M_n \supsetneq M_{n-1} \supsetneq M_{n-2} \cdots M_1 \supsetneq M_0 = 0$$

such that for each $i = 1, \cdots, n$, there is some $P_i \in \operatorname{Spec}(R)$ such that $M_i/M_{i-1} \cong R/P_i$. Such a chain of submodules is called a *prime filtration* of M.

The following fact is best proved in commutative algebra.

Theorem 6.37. *Let R be a Noetherian ring and M a finitely generated R-module with prime filtration*

$$M = M_n \supsetneq M_{n-1} \supsetneq M_{n-2} \cdots M_1 \supsetneq M_0 = 0$$

such that for each $i = 1, \cdots, n$, there is some $P_i \in \operatorname{Spec}(R)$ such that $M_i/M_{i-1} \cong R/P_i$. Then

1. *the minimal elements with respect to containment of the list $P_1, \ldots, P_n$ are the minimal primes of M*
2. *for each minimal prime of M, the number of times it appears in the list $P_1, \ldots, P_n$ is equal to $\ell_{R_P}(M_P)$, independent of the choice of filtration.*

We prove a theorem relating the multiplicity of a module to the length of its localizations.

Theorem 6.38 (Associativity (or additivity) formula for multiplicity)**.** *Let R be a Noetherian graded ring and M a finitely generated graded R-module. Then*

$$e(M) = \sum_{P \in \operatorname{Min}(M), \dim(R/P) = \dim(M)} \ell_{R_P}(M_P) e(R/P).$$

Proof. Consider a prime filtration of M

$$M = M_n \supsetneq M_{n-1} \supsetneq M_{n-2} \cdots M_1 \supsetneq M_0 = 0.$$

It breaks up into short exact sequences

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \cong R/P_i \to 0.$$

The Hilbert polynomial behaves just like the Hilbert function (see Lemma 6.7) so we have

$$P_{M_i} = P_{M_{i-1}} + P_{R/P_i} \text{ for each } 1 \leq i \leq n.$$

Adding all these identities we conclude

$$P_M = \sum_{i=1}^{n} P_{R/P_i}.$$

To compute $e(M)$ we determine the leading coefficient of the sum $\sum_{i=1}^n P_{R/P_i}$. Since the degree of this polynomial is $\dim(M) - 1$ we may ignore all the summands which have $\dim(R/P_i) < \dim(M)$ because their Hilbert polynomials have lower degree. Each $P \in \mathrm{Min}(M)$ such that $\dim(R/P) = \dim(M)$ appears $\ell_{R_P}(M_P)$ times in the sum. So the leading coefficient of P_M is the same as the leading coefficient of

$$\sum_{P \in \mathrm{Min}(M), \dim(R/P) = \dim(M)} \ell_{R_P}(M_P) P_{R/P},$$

which yields the desired identity. □

We are now ready to prove an improved version of Bezout's theorem.

Theorem 6.39 (Bezout's Theorem - with multiplicities)**.** *Let $X \subseteq \mathbb{P}_k^n$ be a projective variety of dimension ≥ 1 and let $H = V^{\mathbb{P}}(f)$ be a hypersurface that does not contain any irreducible component of X. Let $Z_1, \ldots, Z_s$ be the irreducible components of $X \cap H$ of highest dimension. Then for $R = k[X_0, \ldots, X_n]$ we have*

$$\sum_{i=1}^s \ell\left(\frac{R}{I(X) + (f)}\right)_{I(Z_i)} \deg(Z_i) = \deg X \cdot \deg H.$$

In particular, $\sum_{i=1}^s \deg(Z_i) \leq \deg X \cdot \deg H$.

Proof. We have that H does not contain any irreducible component of X if and only if $f \notin \bigcup_{P \in \mathrm{Ass}(R/I(X))} P$ if and only if f is not a zero-divisor on $R/I(X)$. There is an exact sequence

$$0 \to R/I(X) \xrightarrow{\cdot f} R/I(X) \to R/(I(X) + (f)) \to 0$$

which yields

$$H_{R/I(X)} \cdot (1 - t^{\deg(f)}) = H_{R/(I(X)+(f))}.$$

If we have in reduced form

$$H_{R/I(X)} = \frac{h(t)}{(1-t)^d}$$

then

$$H_{R/(I(X)+(f))} = \frac{h(t)(1 - t^{\deg(f)})}{(1-t)^d} = \frac{h(t)(1 + t + \cdots + t^{\deg(f)-1})}{(1-t)^{d-1}}$$

so $e(R/(I(X) + (f))) = h(1) \cdot \deg(f) = \deg X \cdot \deg H$.

Finally, by Theorem 6.38 it follows that

$$e(R/(I(X) + (f))) = \sum_{i=1}^s \ell\left(\frac{R}{I(X) + (f)}\right)_{I(Z_i)} \deg(Z_i).$$

This finishes the proof. □

Bezout's original theorem was the particular case of Theorem 6.39 where X and H are both curves in $\mathbb{P}^2$. It states that

> Two curves of degrees d_1 and d_2 in $\mathbb{P}^2$ that have no common irreducible component intersect in at most $d_1 \cdot d_2$ points (or exactly $d_1 \cdot d_2$ points counted with multiplicities).

The following contrapositive of Bezout's theorem is frequently useful.

Corollary 6.40. *If two curves of degrees d_1 and d_2 in $\mathbb{P}^2$ intersect in more than $d_1 \cdot d_2$ points (counting multiplicities) then the curves must have a common irreducible component.*

Example 6.41. Unfortunately the formula in Theorem 6.39 no longer holds if we try to replace the hypersurface H with an arbitrary projective variety Y. Consider $R = k[x, y, z, w, t]$ and put

$$X = V^{\mathbb{P}}((x, z) \cap (y, w)) \text{ and } Y = V^{\mathbb{P}}(x - y, z - w) \subseteq \mathbb{P}^4.$$

Then $X \cap Y$ is the point $\mathbf{a} = [0 : 0 : 0 : 0 : 1]$ which should have multiplicity 2 as $\deg X = 2$ and $\deg Y = 1$. But

$$\ell\left(\frac{R}{I(X) + I(Y)}\right)_{I([\mathbf{a}])} = e\left(\frac{R}{I(X) + I(Y)}\right) = e\left(\frac{k[x, y, t]}{(x^2, xy, y^2)}\right) = 3.$$

Chapter 7

Divisors

Monday, April 28 2025

7.1 Weil divisors and principal divisors

Divisors are a means towards understanding maps from a variety to projective space.

Recall that if X is an irreducible affine variety the field of *rational functions* on X is the fraction field of the coordinate ring

$$k(X) = \left\{ \frac{f}{g} \mid f, g \in k[X], g \neq 0 \right\}.$$

If X is an irreducible projective variety then $k(X)$ is the *homogeneous localization* of the coordinate ring of X at (0) that is

$$k(X) = \left\{ \frac{f}{g} \mid f, g \in k[X] \text{ homogeneous of the same degree}, g \neq 0 \right\}.$$

Example 7.1 (The divisor of zeros and poles)**.** Let $X = \mathbb{A}^1_k$. Then every rational function on $\mathbb{A}^1_k$ has the form

$$\varphi = \frac{c(x - z_1)^{a_1} \cdots (x - z_r)^{a_r}}{(x - p_1)^{b_1} \cdots (x - p_s)^{b_s}} \in k(A^1_k) = k(x)$$

for some $c, z_i, p_i \in k$ and $a_i, b_i \in \mathbb{N}$. We call the z_i's the *zeros* of φ and the p_i's the *poles* of φ. Then we may encode φ as the following formal sum

$$\operatorname{div}(\varphi) = a_1\{z_1\} + \cdots + a_r\{z_r\} - b_1\{p_1\} - \cdots - b_s\{p_s\}$$

Note that the sets $\{z_1\}, \cdots, \{z_r\}, \{p_1\}, \cdots, \{p_s\}$ are codimension 1 subvarieties of $\mathbb{A}^1_k$.

Now we generalize this.

Definition 7.2. A *prime divisor* on a variety X is a codimension 1 irreducible subvariety of X. A *Weil divisor* D on X is a formal $\mathbb{Z}$-linear combination of prime divisors

$$D = \sum_{i=1}^{t} k_i D_i \text{ with } k_i \in \mathbb{Z}.$$

The *support* of D is the list of prime divisors occurring in D with non-zero coefficient. The set of all divisors on X form a group $\operatorname{Div}(X)$, the free abelian group on the set of prime divisors of X. The zero element is the trivial divisor $D = \sum 0 D_i$ with $\operatorname{Supp}(0) = \emptyset$.

Example 7.3. In $\mathbb{P}^2$, here are some prime divisors: $C = V(xy - z^2), L_1 = V(x), L_2 = V(y)$. Here are some divisors which are not prime: $2C, 2L_1 - L_2$.

Example 7.4 (The divisor of zeros and poles)**.** Let $X = \mathbb{A}^n_k$. Then every rational function on $\mathbb{A}^n_k$ has the form

$$\varphi = \frac{f_1^{a_1} \cdots f_r^{a_r}}{g_1^{b_1} \cdots g_s^{b_s}} \in k(\mathbb{A}^n_k)$$

for some irreducible polynomials $f_i, g_i \in k[\mathbb{A}^n_k]$ and $a_i, b_i \in \mathbb{N}$. Then the divisor of φ is

$$\operatorname{div}(\varphi) = a_1 V(f_1) + \cdots + a_r V(f_r) - b_1 V(g_1) - \cdots - b_s V(g_s).$$

All divisors in $\mathbb{A}^n_k$ are of this form.

In general, on almost any variety X we want to associate to each $\varphi \in k(X) \setminus \{0\}$ some divisor, $\operatorname{div}(\varphi)$ in a similar way to the divisor of zeros and poles in Example 7.1, in such a way that the map

$$k(X)^* = k(X) \setminus \{0\} \to \operatorname{Div}(X) \qquad \varphi \mapsto \operatorname{div}\varphi = \sum_{D \subseteq X \text{prime}} \operatorname{ord}_D(\varphi) \cdot D$$

preserves the group structure on $k(X)^*$, that is, $\operatorname{div}(\varphi_1\varphi_2) = \operatorname{div}(\varphi_1) + \operatorname{div}(\varphi_2)$. Here by $\operatorname{ord}_D(\varphi)$ we mean the "order of vanishing" of φ along D, which will be defined later.

Definition 7.5. Suppose that X is a quasi-projective variety and Y is a subvariety of X. Then the local ring $\mathcal{O}_{X,Y}$ is defined to be the localization $\mathcal{O}_{X,Y} = k[U]_{I_U(Y)}$, where U is any affine open subset of X such that $Y \cap U \neq \emptyset$.

Remark 7.6. The ring $\mathcal{O}_{X,Y}$ does not depend on a choice of open set.

Definition 7.7. Suppose that R is a local Noetherian domain of dimension 1 with fraction field K. We define the order of vanishing along R to be

$$\operatorname{ord}_R : K^* \to \mathbb{Z} \qquad \operatorname{ord}_R(f) = \ell_R(R/(f)) \text{ if } f \in R \setminus \{0\}$$

and we set $\operatorname{ord}_R(f/g) = \operatorname{ord}_R(f) - \operatorname{ord}_R(g)$ for $f, g \in R$ both nonzero.

Lemma 7.8. *In the setup of the above definition we have for $f, g \in K^*$*

$$\mathrm{ord}_R(fg) \quad = \quad \mathrm{ord}_R(f) + \mathrm{ord}_R(g)$$

Proof. It suffices to prove the first equality for $f, g \in R$ both nonzero. This follows from the short exact sequence

$$0 \to R/(f) \xrightarrow{\cdot g} R/(fg) \to R/(g) \to 0$$

since length is additive on short exact sequences. □

Definition 7.9. Let X be an irreducible quasi-projective variety and let $\varphi \in k(X)^*$. The *order of vanishing* of φ along a prime divisor D is $\mathrm{ord}_D(\varphi) := \mathrm{ord}_{\mathcal{O}_{X,D}}(\varphi)$.

Definition 7.10. Let X be an irreducible quasi-projective variety and let $\varphi \in k(X)^*$. The *principal Weil divisor* associated to φ is the Weil divisor

$$\mathrm{div}(\varphi) = \sum_{D \subseteq X \text{prime}} \mathrm{ord}_D(\varphi) \cdot D$$

Theorem 7.11. *Let X be an irreducible quasi-projective variety. The sum defining* $\mathrm{div}(\varphi)$ *in Definition 7.10 is finite. Moreover the function* $k(X)^* = k(X) \setminus \{0\} \to \mathrm{Div}(X)$ $\varphi \mapsto \mathrm{div}\,\varphi$ *is a group homomorphism.*

Proof. Since every quasi-projective variety has an open cover by a finite number of affine open sets, we reduce to the case when X is affine. Write $\varphi = \frac{f}{g}$ where $f, g \in k[X]$. Now $V_X(f)$ has only a finite number of irreducible components, and $\mathrm{ord}_D(f) = 0$ unless D is an irreducible component of $V_X(f)$. The same statement holds for g, and since $\mathrm{ord}_D(\varphi) = \mathrm{ord}_D(f) - \mathrm{ord}_D(g)$, we have that the only D that can appear in the sum are irreducible components of $V_X(f)$ or of $V_X(g)$.

The fact that $\mathrm{div}(\varphi_1\varphi_2) = \mathrm{div}(\varphi_1) + \mathrm{div}(\varphi_2)$ follows from Lemma 7.8. □

Wednesday, April 30

One problem with our Definition 7.9 of order is that we would like $\mathrm{ord}_D(\varphi)$ to be positive if and only if φ is a regular function. But this need not be so.

Example 7.12. Consider $X = U = V(x^3 - y^2)$, $\varphi = y/x$ and $D = \{(0,0)\}$. Then

$$\mathrm{ord}_D(y/x) = \mathrm{ord}_D(y) - \mathrm{ord}_D(x) = 3 - 2 = 1$$

but y/x is not a regular function on X. (However it is the "square root" of the regular function x as $(y/x)^2 = x$ in $k[X]$.

The problem here turns out to be that the coordinate ring of the above curve is not *normal*, that is, integrally closed in its coordinate field. Normal rings of dimension one are very nice.

Definition 7.13. A *discrete valuation ring* (DVR) is a Noetherian local domain R with any of the following equivalent properties:

1. R is regular of dimension 1.
2. R is normal (integrally closed in its fraction field) of dimension 1.
3. R is a UFD with one irreducible element, π.
4. The maximal ideal of R is principal, generated by a *uniformizer* π.
5. Every nonzero ideal of R is (π^t) for some $t \in \mathbb{N}$.

Definition 7.14 (Valuation)**.** Suppose that K is a field. A *valuation* ν of K is a map $\nu : K^* \to G$ to a totally ordered group G such that

1. $\nu(fg) = \nu(f) + \nu(g)$ for all $f, g \in K^*$,
2. $\nu(f + g) \geq \min\{\nu(f), \nu(g)\}$ for all $f, g \in K^*$.

The *valuation ring* of ν is $V = \{f \in K^* \mid \nu(f) \geq 0\} \cup \{0\}$.

Definition 7.15. If R is a DVR with uniformizer π then any $f \in R$ can be written as $f = \text{unit} \cdot \pi^r$ for some $r \in \mathbb{N}$. We define a valuation $\nu_R(f) = r$. More formally

$$\nu_R(f) = \min\{t \mid f \in (\pi)^t\}.$$

We extend this by

$$\nu_R\left(\frac{f}{g}\right) = \nu(f) - \nu(g), \text{ equivalently } \frac{f}{g} = \text{unit} \cdot \pi^{\nu\left(\frac{f}{g}\right)}.$$

Clearly the elements of positive valuation are exactly the non-zero elements of R.

Notice that $\nu_R(f) = \text{ord}_R(f)$ as for $f = \text{unit} \cdot \pi^r$ we have $\ell(R/(f)) = \ell(R/(\pi^r)) = r$ with composition series $0 = (\pi^r) \subsetneq (\pi^{r-1}) \subsetneq \cdots \subsetneq (\pi^1) \subsetneq (\pi^0) = R/(f)$.

Definition 7.16. Let X be a quasi-projective variety. We say X is *normal* if any of the following equivalent conditions hold:

1. For all points $\mathbf{a} \in X$, the local ring $\mathcal{O}_{X,\mathbf{a}}$ is normal.
2. For all subvarieties $Y \subseteq X$, $\mathcal{O}_{Y,X}$ is normal.
3. For every open affine $U \subseteq X$, $k[U] = \mathcal{O}_X(U)$ is normal.

Warning: most sources only define Weil divisors in normal varieties. If X is a normal variety we can rewrite the definition of a principal divisor as follows

$$\text{div}(\varphi) = \sum_{D \subseteq X \text{prime}} \nu_D(\varphi) \cdot D,$$

where $\nu_D(\varphi) = \nu_{\mathcal{O}_{X,D}}(\varphi)$ is the valuation described in Definition 7.15.

We see this coincides with Examples 7.1, 7.4 as $\mathbb{A}^n_k$ is normal and we have for any $f \in k[x_1, \ldots x_n]$ with irreducible factorization $f = f_1^{a_1} \cdots f_r^{a_r}$ that in the DVR $\mathcal{O}_{\mathbb{A}^n_k, V(f_i)} = k[x_1, \ldots, x_n]_{(f_i)}$, the element $f = \textit{unit} \cdot f_i^{a_i}$ so $\nu_{V(f_i)} f = a_i$.

Definition 7.17. We say a Weil divisor $\sum_{i=1}^t k_i D_i$ with D_i prime divisors is *effective* if each $k_i \geq 0$. The set of effective divisors is denoted $\text{Eff}(X)$ and it is a cone (closed under taking $\mathbb{N}$-linear combinations).

Theorem 7.18. *Let φ be a nonzero rational function on an irreducible normal variety X. Then φ is regular on X if and only if $\operatorname{div}\varphi$ is effective.*

Proof. Say $\varphi \in k(X)^*$ and $\operatorname{div}\varphi$ is effective. It suffices to check $\varphi|_U$, where U is affine open in X, is regular. On U, we have that $k[U]$ is normal and $\varphi \in k(U) = k(X)$ satisfies $\operatorname{ord}_D(\varphi) \geq 0$ for all prime divisors D that meet U. It follows that $\nu_{\mathcal{O}_{D,X}} \geq 0$ and so φ belongs to in the DVR $\mathcal{O}_{D,X}$. Then

$$\varphi \in \bigcap_{D \text{ prime in } U} \mathcal{O}_{D,X} = k[U]$$

because a normal ring is the intersection of its localizations at height 1 primes. □

7.2 The divisor class group

We have established in Theorem 7.11 that there is a group homomorphism $k(X)^* \to \operatorname{Div}(X), \varphi \mapsto \operatorname{div}(\varphi)$. The image of this homomorphism is the subgroup of *principal divisors* of X, denoted $P(X)$.

Definition 7.19. The *class group* of an irreducible variety X is the quotient group

$$\operatorname{Cl}(X) = \frac{\operatorname{Div}(X)}{P(X)}.$$

As $\operatorname{Div}(X)$ is an abelian group, so is $\operatorname{Cl}(X)$. For $D, D' \in \operatorname{Div}(X)$ we write $D \sim D'$ if $D - D' \in P(X)$, equivalently D and D' are equal in $\operatorname{Cl}(X)$. This is an equivalence relation called *linear equivalence* of divisors.

Example 7.20 (Class group of $\mathbb{A}_k^n$)**.** $\operatorname{Cl}(\mathbb{A}_k^n) = 0$

Any prime divisor D is given as $D = V(P)$ where P is a prime ideal of height one. Take $0 \neq f \in P$. Since P is prime some irreducible factor of f must be in P, so we may assume f is irreducible and so (f) is prime. Since $(0) \subsetneq (f) \subseteq P$ and $\operatorname{ht} P = 1$ we conclude that $P = (f)$. Thus any prime divisor is the vanishing set of a principal ideal.

Now take an arbitrary divisor $\sum_{i=1}^r a_i V(f_i) - (\sum_{i=1}^s b_i V(g_i))$ for some f_i, g_i irreducible elements and a_i, b_i non-negative integers. Note that

$$\operatorname{div}\left(\frac{f_1^{a_1} \cdots f_r^{a_r}}{g_1^{b_1} \cdots g_s^{b_s}}\right) = \sum_{i=1}^r a_i V(f_i) - (\sum_{i=1}^s b_i V(g_i))$$

so $\operatorname{Div}(\mathbb{A}_k^n) = P(\mathbb{A}_k^n)$ and thus $\operatorname{Cl}(\mathbb{A}_k^n) = 0$.

Remark 7.21. The same argument as above shows that if X is an affine variety such that $k[X]$ is a UFD, then $\operatorname{Cl}(X) = 0$.

Example 7.22 (Class group of $\mathbb{P}_k^n$)**.** $\mathrm{Cl}(\mathbb{P}_k^n) \cong \mathbb{Z}$ is generated by the equivalence class of a hyperplane.

Claim 7.23. There is a surjective group homomorphism

$$\deg : \mathrm{Div}(\mathbb{P}_k^n) \to \mathbb{Z}, \qquad \deg\left(\sum_{i=1}^{s} k_i D_i\right) = \sum_{i=1}^{s} k_i \deg(D_i),$$

where $\deg(D_i)$ is the degree of a prime divisor D_i as a projective variety. This map is a group homomorphism by definition.

By an argument similar to the previous example, each prime divisor D in $\mathbb{P}_k^n$ is given by $D = V(F)$ for some irreducible homogeneous polynomial $F \in k[X_0, \ldots, X_n]$. Then we know $\deg(D) = \deg(F)$ by Example 6.16. In particular if H is a hyperplane this gives $\deg(H) = 1$. Now $\deg(nH) = n \cdot 1 = n$ for any $n \in \mathbb{Z}$ and so the map above is surjective.

We compute the kernel of the above map. Say $\sum_{i=1}^{r} a_i V(F_i) - (\sum_{i=1}^{s} b_i V(G_i)) \in \mathrm{Ker}(\deg)$. Then $\sum_{i=1}^{r} a_i \deg(F_i) - (\sum_{i=1}^{s} b_i \deg(G_i)) = 0$ and so

$$\varphi = \frac{F_1^{a_1} \cdots F_r^{a_r}}{G_1^{b_1} \cdots G_s^{b_s}} \in k(X)^*$$

because the numerator and denominator are homogeneous of the same degree. This shows that $\mathrm{Ker}(\deg) = P(\mathbb{P}_k^n)$. By the first isomorphism theorem we deduce that $\mathrm{Cl}(\mathbb{P}_k^n) = \mathrm{Div}(\mathbb{P}_k^n)/P(\mathbb{P}_k^n) \cong \mathbb{Z}$.

The method of the previous example applied to the bihomogeneous coordinate ring of $\mathbb{P}_k^n \times \mathbb{P}_k^m$ proves the next result.

Example 7.24 (Class group of $\mathbb{P}_k^n \times \mathbb{P}_k^m$)**.** $\mathrm{Cl}(\mathbb{P}_k^n \times \mathbb{P}_k^m) \cong \mathbb{Z} \times \mathbb{Z}$ is generated by a prime divisor L_1 of bidegree (1, 0) and a prime divisor L_2 of bidegree (0, 1). The divisor L_1 is equal to $H_1 \times \mathbb{P}_k^m$, where H_1 is a hyperplane in $\mathbb{P}_k^n$, and $L_2 = \mathbb{P}_k^n \times H_2$, where H_2 is a hyperplane in $\mathbb{P}_k^m$.

Note that in both Examples 7.22 and 7.24 the class group is isomorphic to the grading group for the homogeneous coordinate ring. This generalizes to all normal *toric varieties*.

Monday, May 5

Example 7.25. Define an *elliptic curve* to be a degree three smooth, irreducible curve X in $\mathbb{P}_k^2$. Each prime divisor on X is a point, so we can define a map for $a_i \in X, k_i \in \mathbb{Z}$

$$\deg : \mathrm{Div}(X) \to \mathbb{Z}, \qquad \deg\left(\sum_{i=1}^{s} k_i a_i\right) = \sum_{i=1}^{s} k_i.$$

Let $\mathrm{Div}_0(X), P_0(X)$ denote the elements of $\mathrm{Div}(X)$, respectively of $P(X)$ of degree 0. Since $P_0(X)$ is a subgroup of $\mathrm{Div}_0(X)$ we can and set

$$\mathrm{Cl}_0(X) = \frac{\mathrm{Div}_0(X)}{P_0(X)}.$$

Fix a point $a_0 \in X$.

Claim: $\mathrm{Cl}_0(X)$ is generated by the classes of $\{a - a_0 \mid a \in X\}$.

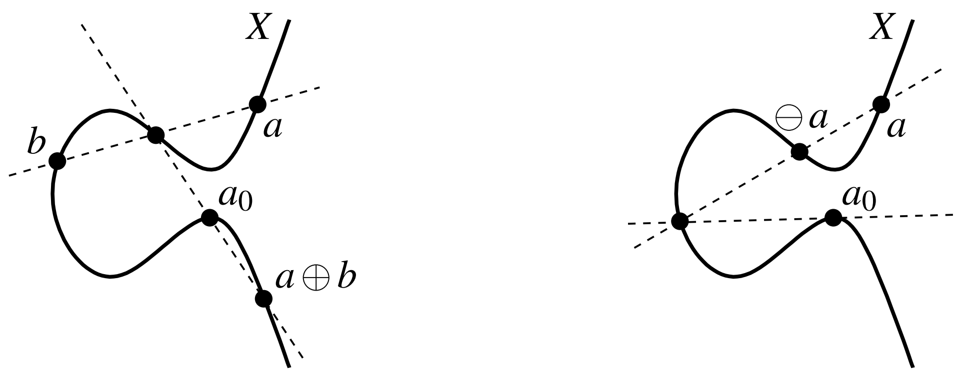
Indeed, any element of $\mathrm{Div}_0(X)$ is of the form $\sum_{i=1}^s k_i a_i$ with $\sum_{i=1}^s k_i = 0$. Then we may rewrite

$$\sum_{i=1}^s k_i a_i = \sum_{i=1}^s k_i a_i - \sum_{i=1}^s k_i a_0 = \sum_{i=1}^s k_i (a_i - a_0).$$

This proves the claim.

We define a binary operations $\oplus$ and a unary operation $\ominus$ on the set of points of X as illustrated in the pictures below. Specifically for two points $a, b \in X$ we consider the unique line L_1 passing through a and b (if $a = b$ then we consider the tangent line to X at a). This line intersects X in a unique third point c by Bezout (since $deg(X) = 3$ and the degree of the line is 1 they intersect in 3 points, counted with multiplicity). The line L_2 through a_0 and c intersects X in a unique third point that we denote $a \oplus b$.

Consider the tangent line to X at a_0 (it is indeed a line, since X is smooth). Since this tangent line intersects X at a_0 with multiplicity two there is another point (could be a_0 as well) where the tangent line intersects X. Call it d. For a point $a \in X$ the line through a and d intersects X at a third point we denote $\ominus a$.



Set $f : X \to \mathrm{Cl}_0(X), f(a) = a - a_0$. We verify that

$$\begin{aligned} f(a \oplus b) &= f(a) + f(b) \\ f(\ominus a) &= -f(a). \end{aligned}$$

Indeed, setting $L_i = V(\ell_i)$ the principal divisors $\mathrm{div}(\ell_1/\ell_2)$ and $\mathrm{div}(\ell_3/\ell_4) \in P_0(X)$ yield the following relations in $\mathrm{Cl}(X)$

$$\begin{aligned} \mathrm{div}(\ell_1/\ell_2) &= a + b + c - (c + a_0 + a \oplus b) \sim 0 \\ \mathrm{div}(\ell_3/\ell_4) &= 2a_0 + d - (a + \ominus a + d) \sim 0. \end{aligned}$$

From the first equation we have

$$a + b \sim a_0 + a \oplus b \text{ or } (a - a_0) + (b - a_0) \sim a \oplus b - a_0$$

and from the last equation we have

$$2a_0 - a + \ominus a \text{ or } -(a - a_0) \sim \ominus a - a_0,$$

as desired.

Claim: the points of X form an abelian group with respect to $\oplus$, where the additive identity is a_0 and the additive inverse is given by $\ominus$. This follows since f preserves the operation and $\mathrm{Cl}(X)$ is an abelian group. In particular, showing associativity holds for $\oplus$ is very hard with the geometric definition. But we can bypass this by applying f and using that $+$ is associative in $\mathrm{Cl}(X)$ instead.

It turns out the map f above is injective (this is non-trivial to show). The first claim showed it is also surjective, Therefore, there is a group isomorphism

$$(X, \oplus) \cong \mathrm{Cl}_0(X).$$

We now want to see how to relate divisor class groups.

Proposition 7.26. *Let X be a normal quasi-projective irreducible variety, Z a proper subvariety of X and $U = Z \setminus X$. Then*

1. *there is a surjective homomorphism*

$$p : \mathrm{Cl}(X) \to \mathrm{Cl}(U), \quad p(\sum_{i=1}^{t} c_i D_i) = \sum_{D_i \cap U \neq \emptyset} c_i (D_i \cap U),$$

where D_i are prime divisors in X

2. *if* $\mathrm{codim}(Z) \geq 2$ *then p is an isomorphism*

3. *if Z is irreducible of* $\mathrm{codim}(Z) = 1$ *then there is an exact sequence*

$$\mathbb{Z} \xrightarrow{1 \mapsto Z} \mathrm{Cl}(X) \xrightarrow{p} \mathrm{Cl}(U) \to 0.$$

Proof. 1. If Y is a prime divisor on X, then $Y \cap U$ is either empty or a prime divisor on U. If $\varphi \in k(X)^*$, and $\mathrm{div}\, \varphi = \sum c_i Y_i$, tnen considering $\varphi|_U$ as a rational function on U, we have $\mathrm{div}\, \varphi|_U = \sum_{Y_i \cap U \neq \emptyset} c_i Y_i$ so indeed we have a well-defined homomorphism $p : \mathrm{Cl}\, X \to \mathrm{Cl}\, U$ as described in the statement.

It is surjective because every prime divisor of U is the intersection of its closure in X with U.

2. The groups $\mathrm{Div}\, X$ and $\mathrm{Cl}\, X$ depend only on subsets of codimension 1, so removing a closed subset Z of codimension 2 doesn't change anything.

3. The kernel of p consists of divisors whose support is contained in Z. If Z is irreducible, the kernel is just the subgroup of $\mathrm{Cl}\, X$ generated by $1 \cdot Z$. □

Example 7.27. Let Z be an irreducible hypersurface of degree d in $\mathbb{P}^n$. Then $\mathrm{Cl}(\mathbb{P}^n_k \setminus Z) \cong \mathbb{Z}/d$

Indeed, from the previous proposition we have an exact sequence

$$\mathbb{Z} \xrightarrow{1 \mapsto Z} \mathrm{Cl}(\mathbb{P}^n_k) \xrightarrow{p} \mathrm{Cl}(U) \to 0.$$

We know from Example 7.22 that $\mathrm{Cl}(\mathbb{P}^n_k) \cong \mathbb{Z}$ and under this isomorphism $Z \mapsto \deg(Z) = d$. So the exact sequence above becomes

$$\mathbb{Z} \xrightarrow{1 \mapsto d} \mathbb{Z} \xrightarrow{p} \mathrm{Cl}(U) \to 0.$$

whence $\mathrm{Cl}(U) \cong \mathbb{Z}/d$.

Wednesday, May 7

Example 7.28. Let B be the blow-up of $\mathbb{P}^2$ at a point $\mathbf{a}$. Then $\mathrm{Cl}(B) \cong \mathbb{Z}H \oplus \mathbb{Z}E$ where H is the class of a hyperplane in $\mathbb{P}^2$ and E is the exceptional set.

First, recall that there is a canonical regular map $\pi : B \to \mathbb{P}^2_k$ and that the exceptional set is $E = \pi^{-1}(\mathbf{a})$. It turns out that E is a prime divisor and in fact $E \cong \mathbb{P}^1$.

The map π induces an isomorphism $\pi|_{B \setminus E} : B \setminus E \to \mathbb{P}^2_k \setminus \{\mathbf{a}\}$, therefore $\mathrm{Cl}(B \setminus E) \cong \mathrm{Cl}(\mathbb{P}^2_k \setminus \{\mathbf{a}\}) = \mathrm{Cl}(\mathbb{P}^2_k) = \mathbb{Z} \cdot H$ where H is any hyperplane in $\mathbb{P}^2_k$. Here we used that $\mathrm{codim}\{\mathbf{a}\} = 2$ and part 2 of Proposition 7.26.

By Proposition 7.26 we have an exact sequence

$$\mathbb{Z} \xrightarrow{1 \mapsto E} \mathrm{Cl}(B) \to \mathrm{Cl}(B \setminus E) \to 0, \text{ that is, } \mathbb{Z} \xrightarrow{1 \mapsto E} \mathrm{Cl}(B) \to \mathbb{Z} \cdot H \to 0.$$

Since the last term is free this sequence splits and we have $\mathrm{Cl}(B) = \mathbb{Z} \cdot E + \mathbb{Z} \cdot H$. We will see that this is a direct sum later.

7.3 27 lines

7.3.1 Intersection product on surfaces

The goal of this section is to set up a basic intersection theory for curves on surfaces. From now on let S be a smooth projective surface. We know that $\mathrm{Cl}(X)$ is an abelian group, but we wish to endow it with a sort of multiplication.

Given two curves $C, D \subset S$, that is, efective divisors on S, we'd like to define the product $C \cdot D$. We require the product to satisfy the following conditions:

$$C \cdot D = \#(C \cap D) \quad \text{provided C and D meet transversely in finitely many points} \tag{7.1}$$

$$(C_1 + C_2) \cdot D = C_1 \cdot D + C_2 \cdot D \quad \text{for all } C_1, C_2, D \in \mathrm{Cl}(S) \tag{7.2}$$

$$C \cdot D = C' \cdot D, C \cdot D = C \cdot D' \quad \text{provided } C \sim C', D \sim D'. \tag{7.3}$$

One can prove (but we won't):

Theorem 7.29. *There is a unique $\mathbb{Z}$-bilinear pairing* $\mathrm{Cl}(S) \times \mathrm{Cl}(S) \to \mathbb{Z}$ *satisfying the above three conditions.*

Example 7.30. Let B be the blow-up of $\mathbb{P}^2_k$ at a point. This is a smooth surface. We know by Example 7.28 that $\mathrm{Cl}(B) = \mathbb{Z} \cdot E + \mathbb{Z} \cdot H$.

We compute $H \cdot E$. We can find a line $H' \subseteq \mathbb{P}^2_k$ that does not pass through the point $\mathbf{a}$ that was blown up. Then $E \cap H' = 0$ and since $H \cong H'$ it follows by (7.3) that $H \cdot E = \emptyset$.

We compute $H \cdot H$. Pick a line $H' \subseteq \mathbb{P}^2_k$ such that H' is not the same line as H. Then $H \cdot H' = 1$ since any two distinct lines in $\mathbb{P}^2_k$ intersect in exactly one point. Since $H \cong H'$ we also have $H^2 = H \cdot H = H \cdot H' = 1$ by (7.3).

Finally we compute $E \cdot E$. Take L, L' to be lines through $\mathbf{a}$. Then $\pi^{-1}(L) = E + D, \pi^{-1}(L') = E + D'$ where D, D' are the strict transforms of L, L'. If we pick L, L' to be distinct lines then D, D' will be disjoint in B. So $D \cdot D' = 0$. Since $L \sim H \sim L'$ we have

$$(H - E) \cdot (H - E) = 0 \iff H^2 - 2H \cdot E + E^2 = 0 \iff 1 - 0 + E^2 = 0 \iff E^2 = -1.$$

This is surprising! We can have negative intersection products.

Now we finish the proof that $\mathrm{Cl}(B) = \mathbb{Z} \cdot E \oplus \mathbb{Z} \cdot H$ (direct sum). Suppose $aH = bE$ for some $a, b \in \mathbb{Z}$. Then

$$a^2 = a^2 H^2 = (aH)^2 = (bE)^2 = b^2 E^2 = -b^2.$$

But if $a^2 = -b^2$ in $\mathbb{Z}$, then we must have $a = b = 0$.

Friday, May 9

7.3.2 27 lines

Earlier we proved that a general cubic surface contains a finite number of lines. Now we show

Theorem 7.31. *A smooth cubic surface contains exactly 27 lines.*

Lemma 7.32. *The blow-up of* $\mathbb{P}^2_k$ *at 6 sufficiently random points is a smooth cubic surface.*

Proof. Say the homogeneous coordinate ring of $\mathbb{P}^2_k$ is $R = k[X_0, X_1, X_2]$. The vector space of homogeneous polynomials of degree 3 in R is 10-dimensional and a typical element has the form

$$F = c_1 X_0^3 + \cdots + c_{10} X_2^3.$$

The condition that F vanishes at a point $\mathbf{a}$ gives a linear equation (in variables $c_1, \ldots, c_{10}$)

$$c_1 a_0^3 + \cdots + c_{10} a_2^3 = 0.$$

The condition that F vanishes at six points then yields a linear system of 6 equations in 10 unknowns. If the points are sufficiently random, the solution space has dimension $10 - 6 = 4$. In other words we have computed that $I(\{p_1, \ldots, p_6)_3 = 4$.

Let F_1, F_2, F_3, F_4 be a basis for the vector space $I(\{p_1, \ldots, p_6)_3 = 4$. We can use this to define a rational (i.e. not everywhere defined) map

$$\phi : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^3_k, \phi(\mathbf{a}) = [F_1(\mathbf{a}) : F_2(\mathbf{a}) : F_3(\mathbf{a}) : F_4(\mathbf{a})]$$

This map is not defined at $V^{\mathbb{P}}(F_1, F_2, F_3, F_4)$, which turns out to be precisely the six points $p_1, \ldots, p_6$.

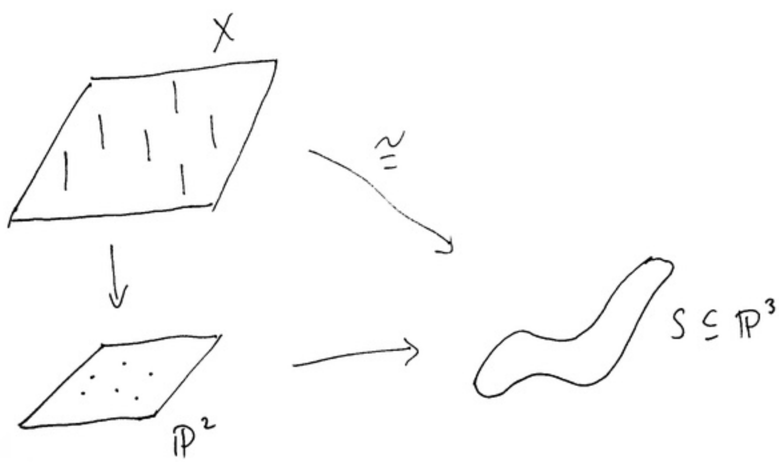
If we construct the blowup $X = \mathrm{Bl}_{p_1,\ldots,p_6}\mathbb{P}^2_k$ then there is a canonical map

$$\pi : X \to \mathbb{P}^2_k$$

and the previous rational map ϕ then extends to a regular map

$$\widetilde{\phi} : X \to \mathbb{P}^3_k$$

The image of $\widetilde{\phi}$: is a surface $S \subseteq \mathbb{P}^3$ isomorphic to X. As X is smooth, so is S.



Claim 7.33. *S is a cubic surface.*

We only need to check that $H^2 = 3$ on S, where H is the class of a hyperplane on $\mathbb{P}^3_k$. Intuitively, H represents a curve on S obtained by intersecting S with a hyperplane, and H^2 represents the number of points on S that we get when we intersect this curve with another hyperplane.

To compute H^2, we work on the surface X. A hyperplane section of S, $H = V^{\mathbb{P}}(aX_0+bX_1+cX_2+dX_3)$ is the image of a cubic curve C in X, specifically $H = \phi(C)$, where $C = \pi^{-1}(aF_1 + bF_2 + cF_3 + dF_4)$. Such a curve has class $C = 3L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6$, where L is the class of a line in $\mathbb{P}^2_k$.

The intersection product on X is as follows:

$$\begin{cases} L^2 = 1 & \\ L \cdot E_i = 0 & \forall 1 \leq i \leq 6 \\ E_i \cdot E_j = 0 & \forall 1 \leq i \neq j \leq 6 \\ E_i^2 = -1 & \forall 1 \leq i \leq 6. \end{cases}$$

It then follows that

$$H^2 = (3L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6)^2 = 3$$

so in fact S is a smooth cubic surface. □

We will take the following converse for granted:

Lemma 7.34. *Any smooth cubic surface can be obtained as the blow-up of $\mathbb{P}^2_k$ at 6 sufficiently random points.*

Lemma 7.35. *There are at least 27 lines on S.*

Proof. How can one recognize a line on S? We see that ℓ is a line on S if and only if $\ell \cdot H = 1$ as $\ell \cdot H = \deg(\ell)$. Say $\phi^{-1}(\ell) = aL + b_1E_1 + b_2E_2 + b_3E_3 + b_4E_4 + b_5E_5 + b_6E_6$ in $\mathrm{Cl}(X)$ and recall that $\widetilde{\phi}^{-1}(H) = 3L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6$. Then

$$\begin{aligned} 1 = \ell \cdot H &= \widetilde{\phi}^{-1}(\ell) \cdot \widetilde{\phi}^{-1}(H) \\ &= (aL + b_1E_1 + b_2E_2 + b_3E_3 + b_4E_4 + b_5E_5 + b_6E_6) \cdot (3L - E_1 - E_2 - E_3 - E_4 - E_5 - E_6) \\ &= 3a + b_1 + b_2 + b_3 + b_4 + b_5 + b_6 \text{ or } \sum_{i=1}^{6} b_i = 1 - 3a. \end{aligned}$$

A few solutions are

$$\begin{aligned} \widetilde{\phi}^{-1}(\ell) &= E_i \\ \widetilde{\phi}^{-1}(\ell) &= L - E_i - E_j \\ \widetilde{\phi}^{-1}(\ell) &= 2L - E_{i_1} - E_{i_2} - E_{i_3} - E_{i_4} - E_{i_5}. \end{aligned}$$

There are 6 solutions of the first kind, $\binom{6}{2} = 15$ solutions of the second kind, 6 solutions of the third kind, for a total of 27 solutions. The curves corresponding to these solutions on X are

- the exceptional divisors $E_1, \ldots, E_6$
- the lines connecting the points p_i and p_j
- the (unique) conic passing through five of the six points $p_1, \ldots, p_6$.

□

It turns out the 27 lines found above are the only lines.

Lemma 7.36. *There are at most 27 lines on S.*

Proof. To justifies this requires knowing a second equation satisfied by $a, b_1, \ldots, b_6$ which is called the *adjunction formula*

$$2g - 2 = \ell(\ell - H), \text{ where } g = 0 \text{ is the genus of } \ell.$$

So $\ell^2 = -1$, that gives $\widetilde{\phi}(\ell)^2 = -1$, i.e,

$$a^2 - b_1^2 - b_2^1 - b_3^2 - b_4^2 - b_5^2 - b_6^2 = 1 \text{ or } \sum_{i=1}^{6} b_i^2 = a^2 - 1.$$

By Cauchy Schwartz, we have $(\sum_{i=1}^{6} b_i)^2 \leq 6 \sum_{i=1}^{6} b_i^2$, which gives $(1-3a)^2 \leq 6(a^2-1)$, and ultimately implies $a < 3$. Once we know that $a \in \{0, 1, 2\}$ it is easy to check that the solutions listed above are the only solutions. □

Index