LEFSCHETZ PROPERTIES THROUGH A TOPOLOGICAL LENSE

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Contents

1. What are these notes about?

These are notes for a graduate summer course on the *Lefschetz Properties* to be held in Krakow, May 6–10, 2024. They are based to a large extent on the monograph [11] by T. Harima, T. Maeno, H. Morita, Y. Numata, A. Wachi and J. Watanabe. This reference contains all the material contained in these notes except for Section 7, which describes more recent developments based on [14]. The treatment of earlier chapters, while deeply influenced by [11], reflects the author's mathematical taste.

The topic of these lectures is the algebraic Lefschetz properties, which are abstractions of the important Hard Lefschetz Theorem from geometry. Section 2 explains the topological context of this result. Section 4 introduces the algebraic Lefschetz properties and their relevance to commutative algebra. Section 5 establishes a correspondence between the strong Lefschetz property and an action of the Lie group $\mathfrak{sl}_2$. Section 6 focuses on the class of Gorenstein rings and their construction via Macaulay's inverse system. Section 5.3 and Section 7 investigate how various constructions of new rings from old interact with the Lefschetz properties.

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2. Cohomology rings and the Hard Lefschetz Theorem

I will give an introduction to the origins of the algebraic Lefschetz properties. The motivation for this topic comes from algebraic topology, so we will spend a bit of time talking about how the Lefschetz property arises there.

2.1. **Cohomology rings.** Let $\mathbb{F}$ be a vector space and let X be a topological space (such as projective space $\mathbb{P}^n$ or the n-dimensional sphere S^n). Let's recall the notion of cohomology of X with coefficients in $\mathbb{F}$.

First, one can think of X as being made out of simple cells (or at least one can approximate X in this manner). This endows X with a *cell complex (CW-complex)* structure.

Example 2.1 (CW structures on sphere)**.** The 2-dimensional sphere S^2 can be obtained from taking a point (0-dimensional cell) and glueing a 2-dimensional disc onto it along its entire boundary. So the CW-structure of S^2 is

$$S^2 = \text{pt} + \text{2-dimensional disc}$$

More generally one can do the same for the n-dimensional sphere S^n:

$$S^n = \text{pt} + \text{n-dimensional disc.}$$

There is another, less economical way to give the sphere a CW-structure. For S^2 one takes two 0-dimensional cells, connects them using two line segments (1-dimensional cells) to form a circle S^1. Then one can glue two 2-dimensional discs via their boundaries to the circle to form S^2. Similarly, there is a CW-structure on S^n with two cells in each dimension summarized by

$$S^n = 2\times\text{pt}+2\times\text{1-dimensional disc}+2\times\text{2-dimensional disc}+\cdots+2\times\text{n-dimensional disc.}$$

Example 2.2 (CW structure on the real projective space)**.** Consider first $\mathbb{P}^n_{\mathbb{R}}$. It can be written as $S^n/\{\pm 1\}$. If we take a CW structure on S^n with two cells in each dimension, with the action of -1 swaps the cells, thus they become identified in the quotient and so $\mathbb{P}^n_{\mathbb{R}}$ has a CW structure with one cell in each dimension.

$$\mathbb{P}^n_{\mathbb{R}} = \text{pt} + \text{1-dimensional cell} + \cdots + \text{n-dimensional cell.}$$

Next consider $\mathbb{P}^n_{\mathbb{C}}$. This has a cell in every even (real) dimension:

$$\mathbb{P}^n_{\mathbb{C}} = \text{pt} + \text{2-dimensional cell} + \cdots + \text{2n-dimensional cell.}$$

Proceeding towards homology, we define a *chain complex* $\mathbf{C}_\bullet(X)$ by letting $C_n(X)$ be the $\mathbb{F}$-vector space generated by the n-dimensional cells of X. There are so-called boundary maps[1], which fit into the following sequence

$$\mathbf{C}_\bullet(X) : 0 \leftarrow \mathbb{F}^{\#\text{0-cells}} \leftarrow \mathbb{F}^{\#\text{1-cells}} \leftarrow \cdots \leftarrow \mathbb{F}^{\#\dim(X)\text{-cells}} \leftarrow 0.$$

There is also a dual version called the *cochain complex* of X with coefficients in R

$$\mathbf{C}^\bullet(X) = \operatorname{Hom}(\mathbf{C}_\bullet(X), \mathbb{F}) : 0 \to \mathbb{F}^{\#\text{0-cells}} \xrightarrow{\partial_1} \mathbb{F}^{\#\text{1-cells}} \xrightarrow{\partial_2} \cdots \xrightarrow{\partial_n} \mathbb{F}^{\#\dim(X)\text{-cells}} \to 0.$$

[1] We will not describe the boundary maps here.

Definition 2.3. The *cohomology groups* of X are defined as

$$H^i(X, \mathbb{F}) = H^i(\mathbf{C}^\bullet(X)) = \operatorname{Ker} \partial_i / \operatorname{Im} \partial_{i-1}.$$

Example 2.4. Based on Example 2.1 we have

$$\mathbf{C}^\bullet(S^n) : 0 \to \mathbb{F} \to 0 \to 0 \to \ldots \to \mathbb{F} \to 0$$

$$H^i(S^n, \mathbb{F}) = \begin{cases} \mathbb{F} & i = 0, n \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{C}^\bullet(\mathbb{P}^n_{\mathbb{C}}) : 0 \to \mathbb{F} \to 0 \to \mathbb{F} \to 0 \to \mathbb{F} \to \ldots \to \mathbb{F} \to 0$$

$$H^i(\mathbb{P}^n_{\mathbb{C}}, R) = \begin{cases} \mathbb{F} & i = \text{even} \\ 0 & i = \text{odd} \end{cases}$$

The special property of these cohomology groups that allows us to study them using tools from ring theory is that they can be assembled into a graded ring.

Definition 2.5. The *cohomology ring* of X is

$$H^\bullet(X, \mathbb{F}) = \bigoplus_{i \geq 0} H^i(X, \mathbb{F}).$$

To study multiplication on this ring we need to define a map called the *cup product*

$$H^m(X, \mathbb{F}) \times H^n(X, \mathbb{F}) \to H^{m+n}(X, \mathbb{F}).$$

For this recall the Künneth isomorphism: for two topological spaces X and Y if one of X or Y has torsion-free homology (true since $\mathbb{F}$ is a field) and has finitely many cells in each dimensions, there is an isomorphism $k : H^\bullet(X \times Y, \mathbb{F}) \cong H^\bullet(X, \mathbb{F}) \otimes_{\mathbb{F}} H^\bullet(Y, \mathbb{F})$. The composite with the diagonal map

$$H^\bullet(X, \mathbb{F}) \otimes_{\mathbb{F}} H^\bullet(X, \mathbb{F}) \xrightarrow{\cong} H^\bullet(X \times X, \mathbb{F}) \xrightarrow{\Delta^*} H^\bullet(X, \mathbb{F})$$

defines the cup product by $x \cup y = \Delta^* k(x \otimes y)$. The cup product is not commutative, but it is what we call *graded commutative*: if $x \in H^m(X, \mathbb{F})$ and $|x| = m$ denotes the degree of x, then

$$x \cup y = (-1)^{|x||y|} y \cup x. \tag{2.1}$$

Note that in a graded commutative ring even degree elements commute with all other elements, while odd degree elements anti-commute with other odd degree elements.

Example 2.6 (Homology ring of a sphere)**.** From Example 2.4 we have

$$H^\bullet(S^n, \mathbb{F}) = \mathbb{F} \oplus \mathbb{F}.$$

Set 1 and e to be the generators of $H^0(S^n, \mathbb{F})$ and $H^n(S^n, \mathbb{F})$ as $\mathbb{F}$-vector speces, respectively. Then 1 is the multiplicative identity of the ring $H^\bullet(S^n, \mathbb{F})$ and $e^2 = e \cup e \in H^{2n}(S^n, \mathbb{F}) = 0$, so

$$H^\bullet(S^n, \mathbb{F}) = \mathbb{F}[e]/(e^2) \text{ with } |e| = n.$$

Example 2.7 (Homology ring of a torus)**.** Applying the Künneth formula to the torus $T^n = S^1 \times \cdots \times S^1$ gives for elements $e_1, \ldots, e_n$ with $|e_i| = 1$

$$H^\bullet(T^n, \mathbb{F}) = \mathbb{F}[e_1]/(e_1^2) \otimes_\mathbb{F} \mathbb{F}[e_2]/(e_2^2) \otimes_\mathbb{F} \mathbb{F}[e_n]/(e_n^2) = \bigwedge_\mathbb{F} \langle e_1, \ldots, e_n \rangle.$$

Note that the tensor product above is taken in the category of graded-commutative algebras which implies that $e_i e_j = -e_j e_i$. If the characteristic of $\mathbb{F}$ is not equal to 2 then this implies $e_i^2 = 0$ for all i. The ring above, denoted $\bigwedge_\mathbb{F}\langle e_1, \ldots, e_n\rangle$, is called an *exterior algebra*. As an $\mathbb{F}$-vector space, a basis of the exterior algebra is given by all the square-free monomials in the variables $e_1, \ldots, e_n$.

Example 2.8 (Homology ring of projective plane)**.** From Example 2.4 we have $H^\bullet(\mathbb{P}^n_\mathbb{C}, \mathbb{F}) = \mathbb{F} \oplus \mathbb{F} \oplus \cdots \oplus \mathbb{F}$, with n summands in degrees $0, 2, \ldots, 2n$. Set x to be the generator of $H^2(\mathbb{P}^n_\mathbb{C}, \mathbb{F})$. It turns out similarly to the above example that

$$H^\bullet(\mathbb{P}^n_\mathbb{C}, \mathbb{F}) = \mathbb{F}[x]/(x^{n+1}), \text{ with } |x| = 2.$$

We can apply the Künneth formula to compute

$$\begin{aligned} H^\bullet(\mathbb{P}^{d_1}_\mathbb{C} \times \mathbb{P}^{d_2}_\mathbb{C} \times \cdots \times \mathbb{P}^{d_n}_\mathbb{C}, \mathbb{F}) &\cong \mathbb{F}[x_1]/(x^{d_1+1}) \otimes_\mathbb{F} \mathbb{F}[x_2]/(x^{d_2+1}) \otimes_\mathbb{F} \cdots \otimes_\mathbb{F} \mathbb{F}[x_n]/(x^{d_n+1}) \\ &\cong \mathbb{F}[x_1, \ldots, x_n]/(x_1^{d_1+1}, \ldots, x_n^{d_n+1}), \text{ with } |x_i| = 2. \end{aligned}$$

2.2. **The Hard Lefschetz Theorem.** We now come to the main result that we have been building up to. Let X be an algebraic subvariaty of $P^n_\mathbb{C}$ and let H denote a (general) hyperplane in $P^n_\mathbb{C}$. Then $X \cap H$ is a subvariety of X of real codimension two, and thus by a, standard construction in algebraic geometry represents a cohomology class $L \in H^2(X, \mathbb{R})$ called the class of a a hyperplane section.

Theorem 2.9 (Hard Lefschetz Theorem)**.** *Let X be a smooth irreducible complex projective variety of complex dimension n (real dimension $2n$), $H^\bullet(X) = H^\bullet(X, \mathbb{R})$, and let $L \in H^2(X, \mathbb{R})$ be the class of a a hyperplane section. Then for $0 \le i \le n$ the following maps are isomorphisms*

$$L^i : H^{n-i}(X) \to H^{n+i}(X), \text{ where } L^i(x) = \underbrace{L \cup \cdots \cup L}_{L^i} \cup x.$$

Remark 2.10. The Hard Lefschetz theorem works for $H^\bullet(X, \mathbb{F})$ where $\mathbb{F}$ is any field of characteristic zero, but the conclusion of the theorem is false in positive characteristic.

The theorem above was first stated by Lefschetz in [15], but his proof was not entirely rigorous. Speaking about his work Lefschetz states:

> *"The harpoon of algebraic topology was planted in the body of the whale of algebraic geometry."*

The first complete proof of Theorem 2.9 was given by Hodge [12]. The "standard" proof today uses the representation theory of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and is due to Chern [3]. Lefschetz's original proof was only recently made rigorous by Deligne[5], who extended it to positive characteristic.

Example 2.11 (The Hard Lefschetz theorem in action)**.** For $H^\bullet(P^n_{\mathbb{C}}) = \mathbb{F}[x]/(x^{n+1})$ the class of a hyperplane is $L = x$ (recall that $|x| = 2$) and it gives whenever $i \equiv n \pmod 2$ isomorphisms

$$\begin{aligned} H^{n-i}(P^n_{\mathbb{C}}) = x^{\frac{n-i}{2}}\mathbb{F} &\xrightarrow{\times x^i} H^{n+i}(P^n_{\mathbb{C}}) = x^{\frac{n+i}{2}}\mathbb{F} \\ x^{\frac{n-i}{2}}y &\mapsto x^i(x^{\frac{n-i}{2}}y) = x^{\frac{n+i}{2}}y. \end{aligned}$$

Cohomology rings of n-dimensional complex projective varieties X with coefficients in a field $\mathbb{F}$ satisfy the following properties:

(1) $H^\bullet(X, \mathbb{F})$ is a graded commutative ring in the sense of (2.1). Its even part $A := H^{2\bullet(}(X, \mathbb{F}) = \bigoplus_{i\geq 0} H^{2i}(X, \mathbb{F})$ is a commutative graded ring as defined in the next chapter. We can re-grade this ring by setting $|x| = i$ if $x \in H^{2i}(X, \mathbb{F})$. With this convention $|L| = 1$.
(2) $H^\bullet(X, \mathbb{F})$ and A are finite dimensional $\mathbb{F}$-vector spaces (so A is an artinian ring cf. Definition 3.8)
(3) $H^\bullet(X, \mathbb{F})$ and A satisfiy Poincaré duality (hence A is a Gorenstein ring cf. Proposition 3.12).

The main objective of this course is to extend the Hard Lefschetz theorem (and some weaker versions) to arbitrary rings which may not necessarily be cohomology rings, but still satisfy at least some of the properties above. Thus we are motivated by the following

Question 2.12. *Which commutative graded rings A that are artinian or both artinian and Gorenstein also satisfy the conclusion of the Hard Lefschetz theorem?*

3. Classes of graded rings

From now on all rings will be commutative unless specified otherwise.

3.1. **Artinian algebras.**

Definition 3.1 (Graded ring)**.** A commutative ring A is an ($\mathbb{N}$–)*graded ring* provided it decomposes as

$$A = \bigoplus_{i\geq 0} A_i$$

with A_i abelian groups such that $\forall i, j \in \mathbb{N}$ $A_iA_j \subseteq A_{i+j}$ ($a \in A_i, b \in A_j \Rightarrow ab \in A_{i+j}$).

From now on we restrict to graded rings A with $A_0 = \mathbb{F}$ a field. I will refer to these as $\mathbb{F}$-*algebras*. Note that in particular such A and each of its homogeneous components A_i is an $\mathbb{F}$ vector space.

Example 3.2. $A = \mathbb{F}[x_1, \dots, x_n]$ is the fundamental example of a graded ring with $A_i =$ the set of homogeneous polynomials of degree i. Note that the degree of x_i is allowed to be an arbitrary positive integers.

Exercise 3.3. Show that if A is a commutative, Noetherian graded $\mathbb{F}$-algebra, then $\dim_{\mathbb{F}} A_i$ is finite for each i.

Definition 3.4 (Hilbert function)**.** The *Hilbert function* of a Noetherian graded $\mathbb{F}$-algebra A is the function

$$h_A : \mathbb{N} \to \mathbb{N}, h_A(i) = \dim_{\mathbb{F}} A_i.$$

The *Hilbert series* of A is the power series $H_A(t) = \sum_{i\geq 0} h_A(i)t^i$.

Exercise 3.5. Prove that the Hilbert function of the polynomial ring $R = \mathbb{F}[x_1, \dots, x_n]$ is given by

$$h_R(i) = \binom{n+i-1}{i}, \forall i \geq 0$$

and the Hilbert series is

$$H_R(t) = \frac{1}{(1-t)^n}.$$

Example 3.6. The Hilbert function of the truncated polynomial ring $A = \frac{\mathbb{F}[x_1,\dots,x_n]}{(x_1,\dots,x_n)^d}$ is given by

$$h_A(i) = \begin{cases} \binom{n+i-1}{i} & \text{if } 0 \leq i < d \\ 0 & \text{if } i \geq d. \end{cases}$$

Thus $H_A(t) = \sum_{i=0}^{d-1} \binom{n+i-1}{i} t^i$.

Example 3.7. Consider $\mathbb{F}$ a field and let $A = \mathbb{F}[x,y,z]/(x^2,y^2,z^2)$. Clearly, A is a finite dimensional $\mathbb{F}$-vector space with basis given by the monomials $\{1, x, y, z, xy, yz, xz, xyz\}$. We see that the elements of A have only four possible degrees 0,1,2,3 and moreover

$$\begin{aligned} A_0 &= \mathrm{Span}_{\mathbb{F}}\{1\} \cong \mathbb{F} \Rightarrow h_A(0) = 1 \\ A_1 &= \mathrm{Span}_{\mathbb{F}}\{x,y,z\} \cong \mathbb{F}^3 \Rightarrow h_A(1) = 3 \\ A_2 &= \mathrm{Span}_{\mathbb{F}}\{xy,yz,xz\} \cong \mathbb{F}^3 \Rightarrow h_A(2) = 3 \\ A_3 &= \mathrm{Span}_{\mathbb{F}}\{xyz\} \cong \mathbb{F} \Rightarrow h_A(3) = 1 \\ A_i &= 0, \forall i \geq 4 \Rightarrow h_A(i) = 0, \forall i \geq 4 \end{aligned}$$

Thus $H_A(t) = 1 + 3t + 3t^2 + t^3$.

In Example 3.6 and Example 3.7 the Hilbert series was in fact a polynomial, equivalently the Hilbert function was eventually equal to zero. We now define a class of graded rings which satisfy this property.

Definition 3.8 (Artinian ring)**.** A (local or) graded $\mathbb{F}$-algebra $(A, \mathfrak{m}, \mathbb{F} = A/\mathfrak{m})$ is *artinian* if any of the following equivalent conditions holds.

(a) A is finite dimensional as a $\mathbb{F}$-vector space.
(b) A has Krull dimension zero.
(c) If $\mathfrak{m}$ is the (homogeneous) maximal ideal of A, then $\mathfrak{m}^p = 0$ for some (hence all sufficiently large) $p \geq 1$. If A is graded this can be restated as $A_d = 0$ for sufficiently large d.
(d) A satisfies the *descending chain condition* for ideals.

(e) There exists a descending sequence of ideals

$$A = \mathfrak{a}_0 \supseteq \mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \cdots \supseteq \mathfrak{a}_\ell = 0 \text{ such that } \mathfrak{a}_{i-1}/\mathfrak{a}_i \cong \mathbb{F}.$$

Such a sequence of ideals is called a *composition series*.

Moreover, if $R = \mathbb{F}[x_1, \dots, x_n]$ is a polynomial ring and $A = R/I$ for some homogeneous ideal I of R then the conditions above are also equivalent to.

(f) For each $0 \leq i \leq n$ there is some integer p_i such that $x_i^{p_i} \in I$.

(g) If $\mathbb{F}$ is algebraically closed, another equivalent condition is $\mathbb{V}(I) = \emptyset$.

3.2. Artinian Gorenstein rings and complete intersections.

Definition 3.9 (Socle)**.** For a graded artinian $\mathbb{F}$ algebra the maximal integer d such that $A_d \neq 0$ is called the *maximal socle degree* of A. The *socle* of A is the ideal

$$(0 :_A \mathfrak{m}) = \{x \in A \mid xy = 0, \forall y \in \mathfrak{m}\}$$

and one can see that there is always a containment $A_d \subseteq (0 :_A \mathfrak{m})$, where d denotes the maximal socle degree of A.

Exercise 3.10. Prove the containment $A_d \subseteq (0 :_A \mathfrak{m})$ claimed above.

Definition 3.11 (Artinian Gorenstein ring)**.** A graded $\mathbb{F}$-algebra is *artinian Gorenstein* (AG) if its socle is a one dimensional $\mathbb{F}$-vector space.

An equivalent characterization of AG algebras is given by the following proposition.

Proposition 3.12 (Poincaré duality)**.** *A graded $\mathbb{F}$-algebra A of maximal socle degree d is AG if and only if for each nonzero element a_{soc} of A_d there exists an $\mathbb{F}$-vector space homomorphism $\int_A : A \to \mathbb{F}$ called an* orientation*, satisfying the following properties:*

(1) $\int_A a_{soc} = 1$, that is, the orientation induces an isomorphism $A_d \cong \mathbb{F}$,

(2) for each element $a \in A_i$ there exists a unique element $b \in A_{c-i}$ so that $\int_A ab = 1$.

In Section 7.2 we will use the notation a_{soc} implicitly to mean fixing the unique orientation on A that satisfies $\int_A a_{soc} = 1$.

Example 3.13. Continuing with Example 3.7, the socle is $(0 :_A \mathfrak{m}) = \mathrm{Span}\{xyz\}$, a 1-dimensional $\mathbb{F}$-vector space. This shows that A is Gorenstein. Take the orientation on A to be specified by $\int_A xyz = 1$. We see that the $\mathbb{F}$-basis elements $\{1, x, y, z, xy, yz, xz, xyz\}$ of A form pairs with respect to the given orientation in the following manner

$$\begin{aligned} \int_A 1 \cdot xyz &= 1 \\ \int_A x \cdot yz &= 1 \\ \int_A y \cdot xz &= 1 \\ \int_A z \cdot xy &= 1. \end{aligned}$$

Example 3.14. Another example of an artinian Gorenstein ring is

$$R = \frac{\mathbb{F}[x, y, z]}{(xy, xz, yz, x^2 - y^2, x^2 - z^2)}.$$

Exercise 3.15. Prove that if A is a graded AG algebra of maximum socle degree d then $h_A(i) = h_A(d-i)$ for each $0 \leq i \leq d$. This is usually stated by saying AG algebras have symmetric Hilbert function.

Definition 3.16. A graded *artinian* $\mathbb{F}$-algebra is a *complete intersection* (CI) if $A = R/I$ where $R = \mathbb{F}[x_1, \dots, x_n]$ and $I = (f_1, \dots, f_n)$, that is, I is a homogeneous ideal generated by as many elements as there are variables in R.

Example 3.17. The rings $A = \mathbb{F}[x_1, \dots, x_n]/(x_1^{d_1}, \dots, x_n^{d_n})$, where $d_1, \dots, d_n \geq 1$ are integers, are called *monomial complete intersections*.

Exercise 3.18. Prove that the rings in Example 3.17 are the only artinian Gorenstein rings of the form R/I where $R = \mathbb{F}[x_1, \dots, x_n]$ and I is an ideal generated by monomials.

All CI rings are Gorenstein, but not all Gorenstein rings are CI, as exemplified by the ring in Exercise 6.22.

4. The Lefschetz properties

4.1. Weak Lefschetz property and consequences.

Definition 4.1 (Weak Lefschetz property)**.** Let $A = \bigoplus_{i=0}^{c} A_i$ be a graded artinian $\mathbb{F}$-algebra. We say that A has the ***weak Lefschetz property (WLP)*** if there exists an element $L \in A_1$ such that for $0 \leq i \leq c-1$ each of the multiplication maps

$$\times L : A_i \to A_{i+1}, x \mapsto Lx \text{is either injective or surjective.}$$

We call L with this property a ***weak Lefschetz element***.

Definition 4.2. The ***non-weak Lefschetz locus*** of a graded artinian $\mathbb{F}$-algebra A is the set (more accurately the algebraic set)

$$NLL_w(A) = \{(a_1, \dots, a_n) \in \mathbb{F}^n \mid L = a_1x_1 + \cdots + a_nx_n \text{ not a weak Lefschetz element on } A\}.$$

The WLP says that $\times L$ has the maximum possible rank, which is referred to as *full rank*.

Exercise 4.3 (Equivalent WLP statements)**.** Prove that for an artinian graded $\mathbb{F}$-algebra A the following are equivalent:

(1) $L \in A_1$ is a weak Lefschetz element for A.
(2) For all $0 \leq i \leq c-1$ the map $\times L : A_i \to A_{i+1}$ has rank $\min\{h_A(i), h_A(i+1)\}$.
(3) For all $0 \leq i \leq c-1$ $\dim_{\mathbb{F}}([(L)]_{i+1}) = \min\{h_A(i), h_A(i+1)\}$.
(4) For all $0 \leq i \leq c-1$ $\dim_{\mathbb{F}}([A/(L)]_{i+1}) = \max\{0, h_A(i+1) - h_A(i)\}$.
(5) For all $0 \leq i \leq c-1$ $\dim_{\mathbb{F}}([0 :_A L]_i) = \max\{0, h_A(i) - h_A(i+1)\}$.

Exercise 4.4. Show that the non-weak Lefschetz locus is a Zariski closed set.

Example 4.5. Take $A = \mathbb{C}[x,y]/(x^2,y^2)$ with the standard grading $|x| = |y| = 1$ and $L = x + y$. Then the multiplication map $\times L$ gives the following matrices with respect to the monomial bases $\{1\}, \{x,y\}$ and $\{xy\}$:

map	matrix	rank	inj/ surj
$A_0 \to A_1$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	1	inj
$A_1 \to A_2$	$\begin{bmatrix} 1 & 1 \end{bmatrix}$	1	surj
$A_i \to A_{i+1}, i \geq 2$	$\begin{bmatrix} 0 \end{bmatrix}$	0	surj

We conclude that A has the WLP and $x + y$ is a Lefschetz element on A.

Example 4.6 (Dependence on characteristic)**.** Take $A = \mathbb{F}[x,y,z]/(x^2,y^2,z^2)$ with the standard grading $|x| = |y| = 1$ and $L = ax+by+cz$. Then the multiplication map $\times L$ is represented by the following matrices with respect to the monomial bases 1 for A_0, $\{x,y,z\}$ for A_1, $\{xy,xz,yz\}$ for A_2, and xyz for A_3:

$$\times L : A_0 \to A_1 \qquad M = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \qquad \text{injective unless } a = b = c = 0$$

$$\times L : A_1 \to A_2 \quad M = \begin{bmatrix} b & a & 0 \\ c & 0 & a \\ 0 & c & b \end{bmatrix}, \quad \det(M) = -2abc$$

$$\times L : A_2 \to A_3 \quad M = \begin{bmatrix} a & b & c \end{bmatrix}, \quad \text{surjective unless } a = b = c = 0.$$

The map $\times L : A_1 \to A_2$ has full rank iff $\mathrm{char}(\mathbb{F}) \neq 2$ and $a \neq 0, b \neq 0, c \neq 0$. We conclude that A has the WLP iff $\mathrm{char}(\mathbb{F}) \neq 2$ because in that case e.g. $L = x + y + z$ is a weak Lefschetz element.

The *non-(weak) Lefschetz locus* of A in this example is

$$\begin{aligned} \mathrm{NLL}_w(A) &= \{(a,b,c) \in \mathbb{F}^3 \mid L = ax + by + cz \text{ is not a weak Lefschetz element on } A\} \\ &= V(abc) = \{(a,b,c) \in \mathbb{F}^3 \mid a = 0 \text{ or } b = 0 \text{ or } c = 0\} \\ &= \text{ the union of the three coordinate planes in } \mathbb{F}^3. \end{aligned}$$

Definition 4.7. A sequence of numbers $h_1, \ldots, h_c$ is called *unimodal* if there is an integer j such that

$$h_1 \leq h_2 \leq \cdots \leq h_j \geq h_{j+1} \geq \cdots \geq h_c.$$

Lemma 4.8. *If B is a standard graded $\mathbb{F}$-algebra and $B_j = 0$ for some $j \in \mathbb{N}$ then $B_i = 0$ for all $i \geq j$.*

Proof. B standard graded means that $B = \mathbb{F}[B_1] = \mathbb{F}[x_1,\ldots,x_n]/I$ where $x_1,\ldots,x_n$ are an $\mathbb{F}$-basis for B_1 so $|x_1| = \cdots = |x_n| = 1$ and I is a homogeneous ideal.

Then we see that $B_i = \mathrm{Span}_{\mathbb{F}}\{B_{i-j}B_j\} = \mathrm{Span}_{\mathbb{F}}\{0\} = 0$ for any $i \geq j$. □

Proposition 4.9. *Suppose that A is a standard graded artinian algebra over a field $\mathbb{F}$. If A has the weak Lefschetz property then A has a unimodal Hilbert function.*

Proof. Let j be the smallest integer such that $\dim_{\mathbb{F}} A_j > \dim_{\mathbb{F}} A_{j+1}$ and let L be a Lefschetz element on A. Then $\times L : A_j \to A_{j+1}$ is surjective i.e. $LA_j = A_{j+1}$. Now consider the cokernel $A/(L)$ of the map

$$A \xrightarrow{\times L} A.$$

We have that $(A/(L))_{j+1} = A_{j+1}/LA_j = 0$, so by the previous Lemma $(A/(L))_{i+i} = A_{i-j}\,(A/(L))_{j+1} = 0$ for $i \geq j$. This means that $\times L : A_i \to A_{i+1}$ is surjective for $i \geq j$ and so we have

$$h_0(A) \leq h_1(A) \leq \cdots \leq h_j(A) \geq h_{j+1}(A) \geq h_{j+2}(A) \geq \cdots \geq h_c(A).$$

$\square$

The proof above yields:

Corollary 4.10. *For a standard graded artinian algebra A there exists $j \in \mathbb{N}$ such that the multiplications maps by a weak Lefschetz element $\times L : A_i \to A_{i+1}$ are injective for $i < j$ after which they become surjective for $i \geq j$.*

Example 4.11 (Dependence on grading)**.** Recall from Example 2.18 that the algebra $A = \mathbb{F}[x,y]/(x^2,y^2)$ with $|x| = |y| = 1$ is standard graded and has WLP and notice that the Hilbert function of A, $1,2,1$ is unimodal.

Take $B = \mathbb{C}[x,y]/(x^2,y^2)$ with $|x| = 1, |y| = 3$. Then B is a graded algebra with nonunimodal Hilbert function $1,1,0,1,1$, but x is a weak Lefschetz element on B.

Take $C = \mathbb{C}[x,y]/(x^2,y^2)$ with $|x| = 1, |y| = 2$. Then C has a unimodal Hilbert function $1,1,1,1$ but does not have the WLP.

4.2. **Strong Lefschetz property and consequences.**

Definition 4.12 (Strong Lefschetz property)**.** Let $A = \bigoplus_{i=1}^{c} A_i$ be a graded artinian $\mathbb{F}$-algebra. We say that A has the ***strong Lefschetz property (SLP)*** if there exists an element $L \in A_1$ such that for all $1 \leq d \leq c$ and $0 \leq i \leq c-d$ each of the multiplication maps

$$\times L^d : A_i \to A_{i+d}, x \mapsto L^d x \text{ is either injective or surjective.}$$

. We call L with this property a ***strong Lefschetz element***.

Remark 4.13. An element $L \in A_1$ is strong Lefschetz on A if and only if for all $1 \leq d \leq c$ and $0 \leq i \leq c-d$ the maps $\times L^d : A_i \to A_{i+d}$ have rank $\min\{h_A(i), h_A(d+i)\}$.

Definition 4.14. The ***non-strong Lefschetz locus*** of a graded artinian $\mathbb{F}$-algebra A is the set (more accurately the algebraic set)

$$NLL_s(A) = \{(a_1,\ldots,a_n) \in \mathbb{F}^n \mid L = a_1x_1 + \cdots + a_nx_n \text{ not a strong Lefschetz element on } A\}.$$

Remark 4.15. The non-strong Lefschetz locus is a Zariski closed set.

Remark 4.16 (SLP $\Rightarrow$ WLP). If A satisfies SLP then A satisfies WLP (the $d = 1$ case).

The following exercise shows this implication is not reversible.

Exercise 4.17. Let $\mathbb{F}$ be a field of characteristic zero and let

$$A = \frac{\mathbb{F}[x,y,z]}{(x^3, y^3, z^3, (x+y+z)^3)}.$$

(1) Find the Hilbert function of A.
(2) Prove that A satisfies WLP but not SLP.

Example 4.18 (Dependence on characteristic)**.** Take $A = \mathbb{F}[x,y]/(x^2,y^2)$ with the standard grading $|x| = |y| = 1$ and $L = ax + by$. Then the multiplication map $\times L^2$ gives the following matrices with respect to the monomial bases $\{1\}, \{x,y\}$ and $\{xy\}$:

map	matrix	rank	inj/ surj
$A_0 \to A_2$	$\begin{bmatrix}2ab\end{bmatrix}$	$\begin{cases} 1 & \mathrm{char}(\mathbb{F}) \neq 2 \\ 0 & \mathrm{char}(\mathbb{F}) = 2 \end{cases}$	$\begin{cases} \text{bij} & \mathrm{char}(\mathbb{F}) \neq 2 \\ \text{none} & \mathrm{char}(\mathbb{F}) = 2 \end{cases}$
$A_i \to A_{i+2}, i \geq 1$	$\begin{bmatrix}0\end{bmatrix}$	0	surj

If $\mathrm{char}(\mathbb{F}) \neq 2$ we conclude that A has the SLP and $ax + by$ where $a \neq 0, b \neq 0$ is a Lefschetz element on A. The *non-(strong) Lefschetz locus* is the union of the coordinate axes in $\mathbb{F}^2$

$$NLL_s(A) = V(ab) = \{(a,b) \in \mathbb{F}^2 \mid a = 0 \text{ or } b = 0\}.$$

However A does not have the SLP if $\mathrm{char}(\mathbb{F}) = 2$ so in that case $\mathrm{NLL}_s(A) = \mathbb{F}^2$.

Proposition 4.19. *Let A be a (not necessarily standard) graded artinian $\mathbb{F}$-algebra which satisfies the SLP. Then A has unimodal Hilbert function.*

Proof. Suppose that the Hilbert function of A is not unimodal. Then there are integers $k < l < m$ such that $\dim_{\mathbb{F}} A_k > \dim_F A_l < \dim_{\mathbb{F}} A_m$. Hence the multiplication map $\times L^{m-k} : A_k \to A_m$ cannot have full rank for any linear element $L \in A$ because it is the composition of $\times L^{m-l} : A_l \to A_m$ and $\times L^{l-k} : A_l \to A_k$, each of which have rank strictly less than $\min\{\dim_{\mathbb{F}} A_k, \dim_{\mathbb{F}} A_m\}$. Thus A cannot have the SLP. ☐

Definition 4.20. . Let $A = \bigoplus_{i=1}^{c} A_i$ be a graded artinian $\mathbb{F}$-algebra. We say that A has the ***strong Lefschetz property in the narrow sense (SLPn)*** if there exists an element $L \in A_1$ such that the multiplication maps $\times L^{c-2i} : A_i \to A_{c-i}, x \mapsto L^{c-2i}x$ are bijections for all $0 \leq i \leq \lceil c/2 \rceil$.

Remark 4.21. SLP in the narrow sense is the closest property to the conclusion of the Hard Lefschetz Theorem 2.9.

Definition 4.22. We say that a graded artinian algebra $A = \bigoplus_{i=1}^{c} A_i$ of maximum socle degree c has a *symmetric Hilbert function* if $h_A(i) = h_A(c-i)$ for $0 \leq i \leq \lceil c/2 \rceil$.

Proposition 4.23. *If a graded artinian $\mathbb{F}$-algebra A has the strong Lefschetz property in the narrow sense, then the Hilbert function of A is unimodal and symmetric. Moreover we have the equivalence:*

A has SLP + symmetric Hilbert function $\Leftrightarrow$ A has SLP in the narrow sense.

Proof. ($\Leftarrow$) The fact that SLP in the narrow sense implies symmetric Hilbert function follows from the definition because the bijections give $\dim_F A_i = \dim_F A_{c-i}$.

The fact that SLP in the narrow sense implies SLP can be noticed by considering $\times L^d : A_i \to A_{i+d}$. For each such d, i there exists $j = c - i - d$ such that:

- if $i \geq (c-d)/2$ then $j = c-i-d \leq i$ and $(\times L^d) \circ (\times L^{j-i}) = \times L^{c-2i}$ is a bijection implies that $\times L^d$ is surjective, hence has full rank;
- if $i < (c-d)/2$ then $c-i > d+i$ and $(\times L^{j-i}) \circ (\times L^d) = \times L^{c-2i}$ is a bijection implies that $\times L^d$ is injective, hence full rank;

($\Rightarrow$) The fact that SLP + symmetric Hilbert function implies SLPn is clear from the definitions. $\square$

Example 4.24. The standard graded algebra $\mathbb{F}[x,y]/(x^2, xy, y^a)$ with $a > 3$ has non-symmetric Hilbert function $1, 2, \underbrace{1, \ldots, 1}_{a-2}$. Notice that A has the SLP because $L = x+y$ is a strong Lefschetz element, A does not satisfy SLPn because its Hilbert function is not symmetric.

4.3. **Stanley's Theorem.** The most famous theorem in the area of investigation of the algebraic Lefschetz properties, and also the theorem which started this, is the following:

Theorem 4.25 (Stanley's theorem)**.** *If* $\operatorname{char}(\mathbb{F}) = 0$*, then all monomial complete intersections, i.e.* $\mathbb{F}$*-algebras of the form*

$$A = \frac{\mathbb{F}[x_1, \ldots, x_n]}{(x_1^{d_1}, \ldots, x_n^{d_n})}$$

with $d_1, \ldots, d_n \in \mathbb{N}$ *have the SLP.*

Proof. Recall that $H^\bullet(\mathbb{P}^{d-1}_{\mathbb{C}}, \mathbb{F}) = \mathbb{F}[x]/(x^d)$, so by Künneth we have

$H^\bullet(\mathbb{P}^{d_1-1}_{\mathbb{C}} \times \mathbb{P}^{d_2-1}_{\mathbb{C}} \times \cdots \times \mathbb{P}^{d_n-1}_{\mathbb{C}}, \mathbb{F}) = \mathbb{F}[x]_1/(x_1^{d_1}) \otimes_{\mathbb{F}} \mathbb{F}[x_2]/(x_2^{d_2}) \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathbb{F}[x_n]/(x_n^{d_n}) = A.$

Since $X = \mathbb{P}^{d_1-1}_{\mathbb{C}} \times \mathbb{P}^{d_2-1}_{\mathbb{C}} \times \cdots \times \mathbb{P}^{d_n-1}_{\mathbb{C}}$ is an irreducible complex projective variety, the Hard Lefschetz theorem says that A has SLP in the narrow sense which implies that A has SLP. $\square$

We will give another proof of Stanley's theorem later in these notes.

Exercise 4.26. With help from a computer make conjectures regarding the WLP and SLP for monomial complete intersections in positive characteristics. A characterization is known for SLP, but not for WLP. See [4, 20] for related work.

4.4. **Combinatorial applications.** The following discussion of a spectacular application of SLP is taken from [23].

A *polytope* is a convex body in Euclidean space which is bounded and has finitely many vertices. Let $\mathcal{P}$ be a d-dimensional simplicial convex polytope with f_i i-dimensional faces, $0 < i < d-1$. We call the vector $f(\mathcal{P}) = (f_0, \ldots, f_{d-1})$ the *f-vector* of $\mathcal{P}$.The problem of obtaining information about such vectors goes back to Descartes and Euler. In 1971 McMullen [18] gave a remarkable condition on a vector $(f_0, \ldots, f_{d-1})$ which he conjectured was equivalent to being the f-vector of some polytope.

To describe this condition, define a new vector $h(\mathcal{P}) = (h_0, ..., h_d)$, called the *$h$-vector* of $\mathcal{P}$, by

$$h_i = \sum_{j=0}^{i} \binom{d-j}{d-i} (-1)^{i-j} f_{j-1}$$

where we set $f_{-1} = 1$. McMullen's conditions, though ho did not realize it, turn out to be equivalent to $h_i = h_{d-i}$ for all i together with the existence of a standard graded commutative algebra A with $A_0 = \mathbb{F}$ and $h_i(A) = h_i - h_{i-1}$ for $1 \leq i < \lfloor d/2 \rfloor$.

Stanley [22] constructed from $\mathcal{P}$ a d-dimensional complex projective toric variety $X(\mathcal{P})$ for which $\dim_{\mathbb{C}} H^{2i}(X(\mathcal{P})) = h_i$. Although $X(\mathcal{P})$ need not be smooth, its singularities are sufficiently nice that the hard Lefschetz theorem continues to hold. Namely, $X(\mathcal{P})$ locally looks like $\mathbb{C}^n/G$, where G is a finite group of linear transformations. Taking $A = H^*(X(\mathcal{P}))/(L)$ with degrees scaled by $1/2$, where L is the class of a hyperplane section, the necessity of McMullen's condition follows from Exercise 4.3(4). Sufficiency was proved about the same time by Billera and Lee [2].

5. Lefschetz property via representation theory of $\mathfrak{sl}_2$

5.1. The Lie algebra $\mathfrak{sl}_2$ and its representations.

Some of the exercises in this section are taken from [21].

Throughout this section let $\mathbb{F}$ be an algebraically closed field of characteristic zero.

Definition 5.1. A *Lie algebra* is a vector space $\mathfrak{g}$ equipped with a bilinear operator $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the following two conditions :

- $[x, y] = -[y, x]$
- $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$

The bilinear operator $[-,-]$ is called the bracket product, or simply the *bracket*. The second identity in the definition is called the *Jacobi identity*.

Remark 5.2. Any associative algebra has a Lie algebra structure with the bracket product defined by commutator $[x, y] = xy - yx$. Associativity implies the Jacobi identity.

The set of $n \times n$ matrices $\mathcal{M}_n(\mathbb{F})$ forms a Lie algebra since it is associative. This Lie algebra is denoted by $\mathfrak{gl}_n(\mathbb{F})$.

Definition 5.3. Since the set of matrices of trace zero is closed under the bracket (because $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any matrices A, B), it forms a Lie subalgebra

$$\mathfrak{sl}_n(\mathbb{F}) = \{M \in \mathfrak{gl}_n(\mathbb{F}) \mid \mathrm{tr}(M) = 0\}.$$

Example 5.4 (The Lie algebra $\mathfrak{sl}_2(\mathbb{F})$)**.** In the case where $n = 2$, $\mathfrak{sl}_2(\mathbb{F})$ is three-dimensional, with basis

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

The three elements E, H, F are called the $\mathfrak{sl}_2$*-triple*.

These elements satisfy the following three relations, which we call the fundamental relations:

$$[E, F] = H, \quad [H, E] = 2E, \quad [H, F] = -2F. \tag{5.1}$$

The algebra $\mathfrak{sl}_2(\mathbb{F})$ is completely determined by these relations.

We are interested in representations of $\mathfrak{sl}_2$.

Definition 5.5 (Lie algebra representation)**.** Let V be an $\mathbb{F}$-vector space. Then $\mathrm{End}(V)$ is a Lie algebra with the bracket defined by $[f, g] = f \circ g - g \circ f$. A *representation* of a Lie algebra $\mathfrak{g}$ is vector space V endowed with a Lie algebra homomorphism

$$\rho : \mathfrak{g} \to \mathrm{End}(V),$$

i.e. a vector space homomorphism which satisfies

$$\rho([x, y]) = [\rho(x), \rho(y)].$$

A representation is called *irreducible* if it contains no trivial (nonzero) subrepresentation i.e. if $W \subsetneq V$ is such that $\rho(W) \subseteq W$ then $W = 0$.

In the case of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{F})$, we abuse notation and call the set of elements $\rho(E), \rho(H), \rho(F)$ just E, H, F and say they form an $\mathfrak{sl}_2$*-triple*.

Exercise 5.6. Let $\mathbb{F}[x, y]_d$ be the vector space of homogeneous polynomials of degree d in $\mathbb{F}[x, y]$. Prove that

(1) $E = x\frac{\partial}{\partial y}, H = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, F = y\frac{\partial}{\partial x}$ form an $\mathfrak{sl}_2$-triple.
(2) Prove that the monomial $x^a y^b$ is an eigenvector of H with eigenvalue $a - b \in \mathbb{Z}$. In particular the eigenvalues of H on $\mathbb{F}[x, y]_d$ are $d, d-2, d-4, \ldots, 4-d, 2-d, -d$.
(3) Prove that a basis of $\mathbb{F}[x, y]_d$ is $y^d, E(y^d), E^2(y^d), \ldots, E^d(y^d)$.

Pictorially this can be summarized as

$$\begin{tikzcd}
0 & \mathbb{F}y^d \arrow[l, "\mathrm{F}"'] \arrow[r, bend right, "\mathrm{E}"'] & \mathbb{F}xy^{d-1} \arrow[l, bend right, "\mathrm{F}"'] \arrow[r, bend right, "\mathrm{E}"'] & \cdots \arrow[l, bend right, "\mathrm{F}"'] \arrow[r, bend right, "\mathrm{E}"'] & \mathbb{F}x^{d-1}y \arrow[l, bend right, "\mathrm{F}"'] \arrow[r, bend right, "\mathrm{E}"'] & \mathbb{F}x^d \arrow[l, bend right, "\mathrm{F}"'] \arrow[r, "\mathrm{E}"'] & 0
\end{tikzcd}$$

We will soon see that the vector space in Exercise 5.6 is the basic building block of all other representations of $\mathfrak{sl}_2$.

An important result on Lie algebra representations are:

Theorem 5.7 (Weyl's Theorem)**.** *Any Lie algebra representation decomposes uniquely up to isomorphism as a direct sum of irreducible representations.*

Definition 5.8 (Weight vectors)**.** Let $\rho : \mathfrak{sl}_2(\mathbb{F}) \to \mathrm{End}(V)$ be a representation. The eigenvalues of H are called *weights* and the eigenvectors are called *weight vectors*. In particular an eigenvector u is called a *lowest weight vector* if $Fu = 0$ and is called a *highest weight vector* if $Eu = 0$.

Example 5.9. In the representation introduced in Exercise 5.6 the highest weight vectors are the elements of $\mathbb{F}x^d$ and the lowest weight vectors are the elements of $\mathbb{F}y^d$.

To justify the name of highest weight we state the following theorem:

Theorem 5.10 (Irreducible representations of $\mathfrak{sl}_2$)**.** *Let $\rho : \mathfrak{sl}_2(\mathbb{F}) \to \mathrm{End}(V)$ be an irreducible representation with $\dim(V) = d+1$. Then there exist a basis $\mathcal{B} = \{v_0, \ldots, v_d\}$ for V such that*

(1) each v_i is an eigenvector for H with eigenvalue $-d+2i$, i.e. $Hv_i = (-d+2i)v_i$
(2) $Ev_i = v_{i+1}$ for $i < d$, $Ev_d = 0$

(3) $Fv_i = i(d-i+1)v_{i-1}$ *for* $i > 0$, $Fv_0 = 0$.

In particular, the elements $E, H, F \in M_{d+1}(\mathbb{F})$ *are represented by the matrices*

$$[E]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{5.2}$$

$$[H]_{\mathcal{B}} = \begin{bmatrix} -d & 0 & \cdots & 0 \\ 0 & -d+2 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & d \end{bmatrix}, \tag{5.3}$$

$$[F]_{\mathcal{B}} = \begin{bmatrix} 0 & 1\cdot d & \cdots & 0 & 0 \\ 0 & 0 & 2(d-1) & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & d\cdot 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{5.4}$$

Exercise 5.11. Find a basis that satisfies the properties given by Theorem 5.10 for the representation $\mathbb{F}[x,y]_d$ introduced in Exercise 5.6.

Theorem 5.10 above says in particular that there is only one representation of $\mathfrak{sl}_2$ of dimension $d+1$ (up to isomorphism). A representative for this isomorphism class can be chosen to be the representation $\mathbb{F}[x,y]_d$ in Exercise 5.6. Furthermore any representation of $\mathfrak{sl}_2$ has a basis consisting of weight vectors. This justifies the following:

Definition 5.12. Let V be a representation of $\mathfrak{sl}_2$ and let $W_\lambda(V) = \{v \in V \mid Hv = \lambda v\}$ be the eigenspace corresponding to a weight (eigenvalue) λ for H. Then there is a decomposition $V = \bigoplus_\lambda W_\lambda(V)$ called the *weight space decomposition* of V.

Remark 5.13. If V is an irreducible representation for $\mathfrak{sl}_2$ and $\dim(V) = n+1$ then the weight spaces are the 1-dimensional spaces $W_{-n+2i}(V) = \mathbb{F}v_i$, with v_i as in Theorem 5.10.

Exercise 5.14.
(1) Suppose that V is a representation of $\mathfrak{sl}_2$ and that the eigenvalues of H on V are $2, 1, 1, 0, -1, -1, -2$. Show that the irreducible decomposition of V is $V \cong \mathbb{F}[x,y]_2 \oplus \mathbb{F}[x,y]_1 \oplus \mathbb{F}[x,y]_1$.
(2) Prove that if V is any representation of $\mathfrak{sl}_2$ then its irreducible decomposition is determined by the eigenvalues of H.

Exercise 5.15. Let V be an $\mathfrak{sl}_2$ representation and set $W_k = \{v \in V \mid H(v) = kv\}$.
(1) Show that $\dim_{\mathbb{F}} W_k = \dim_{\mathbb{F}} W_{-k}$.
(2) Prove that $E^k : W_{-k} \to W_k$ is an isomorphism.
(3) Show that $\dim_{\mathbb{F}} W_{k+2} \leq \dim_{\mathbb{F}} W_k$ for all $k \geq 0$, that is, the two sequences

$$\ldots, \dim_{\mathbb{F}} W_4, \dim_{\mathbb{F}} W_2, \dim_{\mathbb{F}} W_0, \dim_{\mathbb{F}} W_{-2}, \dim_{\mathbb{F}} W_{-4}, \ldots$$

$$\ldots, \dim_{\mathbb{F}} W_3, \dim_{\mathbb{F}} W_1, \dim_{\mathbb{F}} W_{-1}, \dim_{\mathbb{F}} W_{-3}, \ldots$$

are unimodal.

5.2. **Weight space decompositions and the narrow sense of SLP.** We now show that there is a close connection between artinian algebras satisfying SLPn and the representations of $\mathfrak{sl}_2$.

Remark 5.16. If A is a graded artinian $\mathbb{F}$-algebra and L is a linear form, then we can view A as a $\mathbb{F}[L]$-module since by the universal mapping property of polynomial rings there exists a well defined ring homomorphism $\mathbb{F}[L] \to A$ which maps $L \mapsto L$. Since $\mathbb{F}[L]$ is a PID and A is a module over it, the structure theorem for modules over PIDs says that there is a module isomorphims

$$A \cong \mathbb{F}[L]/(p_1^{e_1}) \oplus \cdots \oplus F[L]/(p_k^{e_k})$$

where each p_i is a prime element of $\mathbb{F}[L]$ (no free part since A is finite dimensional). Since A is furthermore graded the elementary divisors $p_i^{e_i}$ must be homogeneous elements of $\mathbb{F}[L]$, thus $p_i = L$ for all i. This implies that A decomposes as a direct sum

$$A \cong S^{(1)} \oplus \cdots \oplus S^{(k)}, \text{ with } S^{(i)} \cong \mathbb{F}[L]/(L^{e_i}).$$

The cyclic $\mathbb{F}[L]$ modules $S^{(i)}$ are the *strands* of multiplication by L on A which were introduced in the lectures regarding the Jordan type. This follows because the action of L on $S^{(i)}$ is given by a single Jordan block of size e_i.

Here is the connection between SLPn and the representations of $\mathfrak{sl}_2$:

Corollary 5.17. *The following are equivalent*

1. *S is a cyclic graded $\mathbb{F}[L]$ module i.e. $S \cong \mathbb{F}[L]/(L^d)$ (not necessarily degree preserving isomorphism)*
2. *$S \cong \mathbb{F}[x, y]_{d-1}$ as an irreducible representation of $\mathfrak{sl}_2$ with $Es = Ls$.*

Proof. This follows because both the action of L on S as well as the action of E on $\mathbb{F}[x, y]_{d-1}$ is given by a single Jordan block matrix. Once the basis of S has been fixed to be $1, L, L^2, \dots, L^{d-1}$, the action of H and F can be simply defined to be the one given by the matrices displayed in Theorem 5.10. □

If we put the $\mathfrak{sl}_2$-module structures on the individual strands together we obtain:

Theorem 5.18 (SLPn via weight decomposition)**.** *Let A be a graded artinian algebra of socle degree c and let $L \in A_1$. The following are equivalent*

1. *L is a strong Lefschetz element on A in the narrow sense,*
2. *A is an $\mathfrak{sl}_2(\mathbb{F})$-representation with $E = \times L$ and the weight space decomposition of A coincides with the grading decomposition via* $\text{weight}(v) = 2\deg(v) - c$. *This means that*

$$A = \bigoplus_{i=0}^{c} A_i = \bigoplus_{i=0}^{c} W_{2i-c}(A), \text{ where } A_i = W_{2i-c}(A).$$

Proof sketch. Suppose L is a strong Lefschetz element on A in the narrow sense. We construct an $\mathfrak{sl}_2(\mathbb{F})$ triple in $\mathrm{End}_{\mathbb{F}}(A)$ as follows: let $E = \times L : A \to A$. Consider the Jordan decomposition of A with respect to the endomorphism E written as $A = \bigoplus V_i$. For each V_i, let $F_i, H_i : V_i \to V_i$ to be the endomorphisms of V_i given with respect to the basis in which $E\big|_{V_i}$ is in Jordan form by the matrices in (5.3) and (5.4), respectively, where $d = \dim(V_i) - 1$. Setting $H = \bigoplus H_i$ and $F = \bigoplus F_i$, one can check E, H, F is an $\mathfrak{sl}_2(\mathbb{F})$ triple. Furthermore, from the lectures on Jordan type one knows that the Jordan blocks of L are centered around the middle degree of A. It follows that if v is an eigenvector of weight $2k-d$ it is in degree $(c-d)/2+k$ (note that $c \equiv d \pmod 2$). Substituting $i = (c-d)/2+k$ it follows that $W_{2i-c}(A) = A_i$.

Conversely, suppose A is an $\mathfrak{sl}_2(\mathbb{F})$-representation with $E = \times L$. Then one can use the information about the grading to verify that the Jordan blocks are centered around degree $\lfloor c/2 \rfloor$. Thus the Jordan degree type is the transpose of the Hilbert function of A. By the lectures on Jodan type it follows that L is a strong Lefschetz element on A in the narrow sense. ☐

5.3. **Tensor products.** From Theorem 5.18, we can deduce how SLPn behaves when we take tensor products. We need the following lemma.

Lemma 5.19. *If $\mathbb{F}$ is an algebraically closed field of characteristic zero A, A' are associative algebras which are representations of $\mathfrak{sl}_2(\mathbb{F})$, then so is $A \otimes_{\mathbb{F}} A'$ with the action $g \cdot (v \otimes v') = (gv) \otimes v' + v \otimes (gv')$. If v, v' are weight vectors then $v \otimes v'$ is also a weight vector with* $\mathrm{weight}(v \otimes v') = \mathrm{weight}(v) + \mathrm{weight}(v')$.

Proof. We show the statement about weights only: say $\mathrm{weight}(v) = \lambda$ and $\mathrm{weight}(v') = \lambda'$ so that $Hv = \lambda v, Hv' = \lambda v'$. Then

$$H(v \otimes v') = (Hv) \otimes v' + v \otimes (Hv') = \lambda v \otimes v' + v \otimes \lambda' v' = (\lambda + \lambda') v \otimes v'$$

shows that $v \otimes v'$ is a weight vector with weight $\lambda + \lambda'$. ☐

Theorem 5.20. *Let $\mathbb{F}$ be an algebraically closed field of characteristic zero. If L is a strong Lefschetz element in the narrow sense on A and if L' is a strong Lefschetz element in the narrow sense on A' then $L \otimes 1 + 1 \otimes L'$ is a strong Lefschetz element in the narrow sense on $A \otimes_{\mathbb{F}} A'$.*

Proof. By Theorem 5.18 we have that if c, c' are the socle degrees of A, A', respectively, then $A_i = W_{2i-c}(A)$ and $A'_j = W_{2j-c'}(A')$, so

$$A = \bigoplus_{i=0}^{c} A_i = \bigoplus_{i=0}^{c} W_{2i-c}(A) \quad \text{and} \quad A' = \bigoplus_{j=0}^{c'} A'_j = \bigoplus_{j=0}^{c'} W_{2j-c'}(A') \tag{5.5}$$

imply

$$A \otimes_{\mathbb{F}} A' = \bigoplus_{i=0,j=0}^{c,c'} A_i \otimes_{\mathbb{F}} A'_j = \bigoplus_{i=0,j=0}^{c,c'} W_{2i-c}(A) \otimes_{\mathbb{F}} W_{2j-c'}(A'). \tag{5.6}$$

From the fact that $\deg(v \otimes v') = \deg(v) + \deg(v')$ and (5.5) we deduce that

$$(A \otimes_{\mathbb{F}} A')_k = \bigoplus_{i=0}^{c} A_i \otimes_F A'_{k-i}.$$

Note that the maximum socle degree of $A \otimes_{\mathbb{F}} A'$ is $c + c'$. From the identity weight$(v \otimes v') = \text{weight}(v) + \text{weight}(v')$ and (5.6) we deduce that

$$W_{2k-c-c'}(A \otimes_{\mathbb{F}} A') = \bigoplus_{i=0}^{c} W_{2i-c}(A) \otimes_{\mathbb{F}} W_{2(k-i)-c'}(A') = \bigoplus_{i=0}^{c} A_i \otimes_{\mathbb{F}} A'_{k-i}.$$

Comparing, we see that $(A \otimes_{\mathbb{F}} A')_k = W_{2k-c-c'}(A \otimes_{\mathbb{F}} A')$, where the weight spaces on $A \otimes_{\mathbb{F}} A'$ correspond to the action

$$E(v \otimes v') = Ev \otimes v' + v \otimes Ev' = Lv \otimes v' + v \otimes L'v' = (L \otimes 1 + 1 \otimes L')v \otimes v'.$$

Theorem 5.18 gives that $L \otimes 1 + 1 \otimes L'$ is a strong Lefschetz element on $A \otimes_{\mathbb{F}} A'$. $\square$

A corollary of Theorem 5.20 is the following

Corollary 5.21 (Tensor product preserves SLPn)**.** *If $\mathbb{F}$ be an algebraically closed field of characteristic zero*[2] *and A, A' are graded artinian $\mathbb{F}$-algebras which satisfy SLPn, then $A \otimes_{\mathbb{F}} A'$ also satisfies SLPn.*

From the above corollary one can easily deduce Stanley's theorem applying induction on the embedding dimension n.

Corollary 5.22 (Stanley's Theorem - second proof)**.** *If $\mathbb{F}$ has characteristic 0, then the algebra $A = \mathbb{F}[x_1, \ldots, x_n]/(x_1^{d_1}, \ldots, x_n^{d_n}) = \mathbb{F}[x_1]/(x_1^{d_1}) \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathbb{F}[x_n]/(x_n^{d_n})$ satisfies SLP in the narrow sense.*

Remark 5.23.

(1) While the symmetric unimodality of Hilbert functions is preserved under taking tensor product, just unimodality is not. For example for

$$A = \mathbb{F}[x, y, z]/(x^2, xy, y^2, xz, yz, z^5)$$

with Hilbert function $1, 3, 1, 1, 1$ we have that the Hilbert function of $A \otimes_{\mathbb{F}} A$ is $1, 6, 11, 8, 9, 8, 3, 2, 1$.

(2) While the SLPn is preserved under taking tensor product, the SLP (not in the narrow sense) is not preserved by tensor product. In the example above A has SLP but since its Hilbert function is not unimodal, $A \otimes_{\mathbb{F}} A$ cannot have the SLP.

The issue in part 2 of the remark is remedied by restricting to Gorenstein algebras, which have symmetric Hilbert function. Recall that for algebras with symmetric Hilbert function the SLP is equivalent to SLPn. Thus we have:

Corollary 5.24. *If $\mathbb{F}$ be an algebraically closed field of characteristic zero*[3] *and A, A' are graded artinian Gorenstein $\mathbb{F}$-algebras which satisfy SLP, then $A \otimes_{\mathbb{F}} A'$ also satisfies SLP.*

[2] This assertion holds true regardless of any assumptions on $\mathbb{F}$, but not by the proof given here.

[3] This assertion holds true regardless of any assumptions on $\mathbb{F}$, but not by the proof given here.

6. Gorenstein rings via Macaulay inverse systems

The description of the dual ring of the polynomial ring in Section 6.1 is taken from [6]. The material in Section 6.2 follows Eisenbud's Commutative Algebra book [7] and Geramita's lectures [8, Lecture 9]. The material on Hessians in Section 6.3 follows [17].

6.1. **The graded dual of the polynomial ring.**

Recall the notion of a dual for an $\mathbb{F}$-vector space:

Definition 6.1. Let V be an $\mathbb{F}$-vector space. Its dual is

$$V^* = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F}) = \{\varphi : V \to \mathbb{F} \mid \varphi \text{ is } \mathbb{F}-\text{linear}\},$$

the vector space of linear functionals on V.

Exercise 6.2. If V is a finite dimensional vector space, there is a natural isomorphism of vector spaces $V \cong V^{**}$.

We extend this idea to construct duals of rings and modules.

Definition 6.3 (Divided power algebra)**.** Say $R = \mathbb{F}[x_1, \dots, x_n]$ is the polynomial ring. Let

$$R^* := \mathrm{Hom}^{\mathrm{gr}}_{\mathbb{F}}(R, \mathbb{F}) = \bigoplus_{i \geq 0} \mathrm{Hom}_{\mathbb{F}}(R_i, \mathbb{F}).$$

We use a standard shorthand for monomials: if $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geq 0}$, then $x^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ is the corresponding monomial in R. If $x^{\mathbf{a}}$ is in R_d, we write $X^{[\mathbf{a}]}$ for the functional (in R^*_d) on R_d which sends $x^{\mathbf{a}}$ to 1 and all other monomials in R_d to 0. We'll make the convention from now on to write X_i for the duals of the elements x_i in R^*_1. As a vector space, R^* is isomorphic to a polynomial ring in the n variables $X_1, \dots, X_n$. However, as we recall shortly, R^* has the structure of a *divided power algebra*. For this reason, we call $X^{[\mathbf{a}]}$ a *divided* monomial and we write $R^* = \mathbb{F}[X_1, \dots, X_n]_{DP}$. Here the notation DP indicates a *divided power algebra*.

The ring R acts on R^* by *contraction*, which we denote by $\bullet$. That is, if $x^{\mathbf{a}}$ is a monomial in R and $X^{[\mathbf{b}]}$ is a divided monomial in R^*, then

$$x^{\mathbf{a}} \bullet X^{[\mathbf{b}]} = \begin{cases} X^{[\mathbf{b}-\mathbf{a}]} & \text{if } \mathbf{b} \geq \mathbf{a}, \\ 0 & \text{otherwise.} \end{cases}$$

This action is extended linearly to all of R and R^*. This action of R on R^* gives a perfect pairing of vector spaces $R_d \times R^*_d \to \mathbb{F}$ for any degree $d \geq 0$. Suppose U is a subspace of R_d. We define

$$U^{\perp} = \{g \in R^*_d : f \bullet g = 0 \text{ for all } f \in U\}.$$

Macaulay [16] introduced the *inverse system* of an ideal I of R to be

$$I^{-1} := \mathrm{Ann}_{R^*}(I) = \{g \in R^* : f \bullet g = 0 \text{ for all } f \in I\}.$$

If I is a homogeneous ideal of R then the inverse system I^{-1} can be constructed degree by degree using the identification $(I^{-1})_d = I_d^{\perp}$. We return to this notion in Definition 6.11.

A priori, R^* is simply a graded R-module. However, R^* can be equipped with a multiplication which makes it into a ring. Suppose $\mathbf{a} = (a_1, \dots, a_n), \mathbf{b} = (b_1, \dots, b_n) \in \mathbb{Z}^n_{\geq 0}$. The multiplication in R^* is defined on monomials by

$$X^{[\mathbf{a}]}X^{[\mathbf{b}]} = \binom{\mathbf{a}+\mathbf{b}}{\mathbf{a}} X^{[\mathbf{a}+\mathbf{b}]}, \tag{6.1}$$

where

$$\mathbf{a}! = \prod_{i=0}^{N} a_i! \quad \text{and} \quad \binom{\mathbf{a}+\mathbf{b}}{\mathbf{a}} = \prod_{i=1}^{n} \binom{a_i+b_i}{a_i}. \tag{6.2}$$

This multiplication is extended linearly to all of R^*. We see from the above definition that if $\mathbf{a} = (a_1, \dots, a_n)$ then $X^{[\mathbf{a}]} = \prod_{i=1}^n X_i^{[a_i]}$.

Exercise 6.4. Now set $X^{\mathbf{a}} = \prod_{i=1}^n X_i^{a_i}$, where the multiplication occurs in the divided power algebra as defined above. Deduce from the above definition that

$$X^{\mathbf{a}} = \mathbf{a}!X^{[\mathbf{a}]}. \tag{6.3}$$

Remark 6.5. In characteristic zero, $\mathbf{a}!$ never vanishes and so, by (6.3), R^* is generated as an algebra by $X_0, \dots, X_N$, just like the polynomial ring. However, in charateristic $p > 0$, R^* is infinitely generated by all the divided power monomials $X_j^{[p^{k_i}]}$ for all $j = 0, \dots, N$ and $k_j \geq 0$. The exercise below justifies this last assertion.

Exercise 6.6. Prove that in characteristic p for any $\mathbf{a} = (a_1, \dots, a_n)$ where $a_j = \sum a_{ij}p^i$, we have

$$X^{[\mathbf{a}]} = \prod_{j=1}^{n}\prod_{i}(X_j^{[p^i]})^{a_{ij}}.$$

Hint: Use Lucas' identity – given base p expansions $a = \sum a_i p^i$ and $b = \sum b_i p^i$ for $a, b \in \mathbb{N}$, then

$$\binom{b}{a} = \prod_{i=0}^{\infty}\binom{b_i}{a_i} \mod p.$$

We now revisit the characteristic zero case. Suppose $\mathbb{F}$ is a field of characteristic zero and let $S = \mathbb{F}[X_1, \dots, X_n]$ be a polynomial ring. Consider the action of R on S by partial differentiation, which we represent by '$\circ$'. That is, if $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geq 0}$, $x^{\mathbf{a}} = x_1^{a_0} \cdots x_n^{a_n}$ is a monomial in R, and $g \in S$, we write

$$x^{\mathbf{a}} \circ g = \frac{\partial^{\mathbf{a}} g}{\partial X^{\mathbf{a}}}$$

for the action of $x^{\mathbf{a}}$ on g (extended linearly to all of R). In particular, if $\mathbf{a} \leq \mathbf{b}$, then

$$x^{\mathbf{a}} \circ X^{\mathbf{b}} = \frac{\mathbf{b}!}{(\mathbf{b}-\mathbf{a})!}X^{\mathbf{b}-\mathbf{a}},$$

where we use the conventions in (6.2). This action gives a perfect pairing $R_d \times S_d \to \mathbb{F}$, and, given a homogeneous ideal $I \subset R$, we define $I_d^{\perp}$ and I^{-1} in the same way as we do for contraction.

Since we are in characteristic zero, the map of rings $\Phi : S \to R^*$ defined by $\Phi(X_i) = X_i$ extends to all monomials via (6.3) to give $\Phi(y^{\mathbf{a}}) = Y^{\mathbf{a}} = \mathbf{a}! Y^{[\mathbf{a}]}$. Thus S and R^* are isomorphic. Moreover, if $F \in R$ and $g \in S$, then $\Phi(F \circ g) = F \bullet \Phi(g)$ [8, Theorem 9.5], so S and R^* are isomorphic as R-modules.

6.2. **Macaulay inverse systems.**

Definition 6.7 (Dualizing functor)**.** Let M be a finitely generated R-module. Define the dual of M to be $D(M) = \mathrm{Hom}_R(M, R^*)$. Let $f : M \to N$ be an R-module homomorphism. Define $D(f)$ to be the induced R-module homomorphism

$$D(f) : D(N) = \mathrm{Hom}_R(N, R^*) \to D(M) = \mathrm{Hom}_R(M, R^*)$$

given by

$$D(f)(\varphi) = \varphi \circ f.$$

This makes D into a contravariant functor in the category of finitely generated R-modules.

Exercise 6.8. Let $A = R/I$ be an artinian $\mathbb{F}$-algebra and let M be a finitely generated A-module. Recall that we defined $D(M) = \mathrm{Hom}_R(M, R^*)$ to be an R-module. This set also has an A-module structure induced from the A-module structure of M, i.e., multiplication by elements of a is given by

$$a\phi(x) = \phi(a \cdot x), \quad \forall a \in A, x \in M.$$

In this exercise we also consider the set $M^* = \mathrm{Hom}_{\mathbb{F}}(M, \mathbb{F})$ with its two structures induced from the R-module structure of M and from the A-module structure of M, respectively, as described in the equation displayed above. Below we show that $D(M) \cong M^*$, so an equivalent way to define the dual module dual to M is M^* (with its R-module structure).

(1) Show that $D(M) \cong \mathrm{Hom}_{\mathbb{F}}(M, \mathbb{F})$ as R-modules.
    Hint: Hom-tensor adjointness may come in handy.
(2) Show that $D(M) \cong \mathrm{Hom}_{\mathbb{F}}(M, \mathbb{F})$ also as A-modules.

We now come to a form of duality that involves the above defined functor.

Theorem 6.9 (Matlis duality)**.** *The functor D induces an anti-equivalence of categories between*

$$\{\textit{noetherian } R\textit{-modules}\} \leftrightarrow \{\textit{artinian } R\textit{-submodules of } R^*\}$$

given by sending $M \mapsto D(M)$.

Next we wish to make the meaning of $D(M)$ more concrete in the special case when $M = R/I$ is a cyclic R-module.

Lemma 6.10. *Suppose I is a homogeneous ideal of a polynomial ring R. We compute*

$$D(R/I) = \mathrm{Hom}_R(R/I, R^*) \cong \mathrm{Ann}_{R^*}(I) = (0 :_{R^*} I) = \{g \in R^* \mid f \bullet g = 0\ \forall f \in I\}.$$

Definition 6.11 (Inverse system)**.** Suppose I is a homogeneous ideal of a polynomial ring R. The *inverse system* of I is the vector space

$$I^{-1} = \{g \in R^* \mid f \bullet g = 0, \forall f \in I\}.$$

Remark 6.12. Don't let the notation deceive you! If I is an ideal of R, it does not mean that $I^{\perp}$ is an ideal (or R^*-submodule) of R^*. It is just an R-module which happens to be a subset of R^*.

Example 6.13. Concretely, say

(1) $I = (x^2, y^3) \subseteq R = \mathbb{F}[x,y]$. Then
$$I^{-1} = (0 :_{R^*} I) = \operatorname{Span}\{X^2, XY, Y^2, X, Y, 1\} = R \bullet XY^2$$
is the R-submodule of R^* generated by XY^2.

(2) $I = (x^2, xy^2, y^3) \subseteq R = \mathbb{F}[x,y]$. Then
$$I^{-1} = (0 :_{R^*} I) = \operatorname{Span}\{XY, Y^2, X, Y, 1\} = R \bullet XY + R \bullet Y^2$$
is an R-submodule of R^* with two generators.

Next, take

Example 6.14. (1) $I = (x) \subseteq R = \mathbb{F}[x,y]$. Then
$$I^{-1} = (0 :_{R^*} I) = \operatorname{Span}\{Y^i \mid i \geq 0\}.$$

(2) $I = (x^d) \subseteq R = \mathbb{F}[x,y]$. Then
$$\begin{aligned} I^{-1} = (0 :_{R^*} I) &= \operatorname{Span}\{X^i Y^J \mid 0 \leq i \leq d-1, j \geq 0\} \\ &= R_0^* \oplus R_1^* \oplus R_2^* \oplus \cdots \oplus R_{d-1}^* \oplus Y R_{d-1}^* \oplus Y^2 R_{d-1}^* \oplus \cdots \oplus Y^k R_{d-1}^* \oplus \cdots \end{aligned}$$

Both of the above $(0 :_{R^*} I)$ are non-finitely generated R-module. We shall see below that this corresponds to R/I not being artinian.

Exercise 6.15. Generalize Example 6.14 to find the inverse system of the ideal defining a point in projective n-space and the inverse systems of all of the powers of this ideal.

We now wish to study the inverse functor involved with in the Matlis duality Theorem 6.9. In order to do this we define the inverse system of an $\mathbb{F}$-subspace of R^*.

Definition 6.16. Let V be an $\mathbb{F}$-vector subspace of the $\mathbb{F}$-algebra R^*. The *inverse system* of V is
$$\operatorname{Ann}_R(V) = \{f \in R \mid f \circ v = 0, \forall v \in V\}.$$
We will be most interested in the case when $V = \operatorname{Span}\{F\}$ is a 1-dimensional $\mathbb{F}$-vector space and thus
$$\operatorname{Ann}_R(F) = \{f \in R \mid f \circ F = 0\}.$$

Macaulay inverse system duality is a concrete version of Matlis duality Theorem 6.9 which can be stated in terms of the inverse systems defined above as follows:

Theorem 6.17 (Macaulay inverse system duality)**.** *With notation as above, there are bijective correspondence between*
$$\begin{aligned} \{R - \textit{modules } M \subseteq R^*\} &\leftrightarrow \{R/I \mid I \subseteq R \textit{ homogeneous ideal}\} \\ M &\mapsto D(M) = R/\operatorname{Ann}_R(M) \\ I^{\perp} = D(R/I) &\leftarrow R/I. \end{aligned}$$

Furthermore, we have the additional correspondences

$$\begin{array}{llcl} (a) & M \textit{ finitely generated} & \iff & R/\operatorname{Ann}_R(M) \textit{ artinian} \\ (b) & M = R \circ F \textit{ cyclic} & \iff & R/\operatorname{Ann}_R(F) \textit{ artinian Gorenstein} \\ & & & \deg(F) = \textit{socle degree of } R/\operatorname{Ann}_R(F). \end{array}$$

The value of Theorem 6.17 often lies in producing examples of artinian Gorenstein rings.

Definition 6.18. In view of statement (b) in Theorem 6.17, the polynomial $F \in R^*$ is called a *Macaulay dual generator* for $R/\operatorname{Ann}_R(F)$.

Example 6.19. The artinian Gorenstein algebra with Macaulay dual generator

$$F = X^2 + Y^2 + Z^2$$

is the ring of Exercise 6.22

$$\mathbb{F}[x,y,z]/\operatorname{Ann}_{\mathbb{F}[x,y,z]}(F) = \mathbb{F}[x,y,z]/(x^2 - y^2, y^2 - z^2, z^2 - x^2, xy, xz, yz).$$

Example 6.20. The artinian Gorenstein algebra with Macaulay dual generator

$$F = X_1^{d_1} \cdots X_n^{d_n}$$

is the monomial complete intersection

$$\mathbb{F}[x_1,\dots,x_n]/\operatorname{Ann}_{\mathbb{F}[x_1,\dots,x_n]}(F) = \mathbb{F}[x_1,\dots,x_n]/(x_1^{d_1+1},\dots,x_n^{d_n+1}).$$

Definition 6.21. For a graded ring A and an integer d, define $A(d)$ to be a the graded ring A with grading modified such that $A(d)_i = A_{d+i}$.

Exercise 6.22. For any homogeneous polynomial $F \in R^*$ of degree d, prove

(1) $\operatorname{Ann}_R(F)^\perp = R \circ F$. This statement is an instance of Macaulay's double annihilator theorem.
(2) the cyclic ring $A = R/\operatorname{Ann}_R(F)$ is artinian Gorenstein if and only if the function $A \to D(A)(-d)$, $a \mapsto [b \mapsto (ab) \circ F]$ is an isomorphism.

Hint for (1): Start by showing the equality is true in degree d, then use the R-module structure.
Hint for (2): Use Proposition 3.12. Prove that the function $a \mapsto (a \bullet F)(0)$ is an orientation on A and that A satisfies Poincaré duality with respect to this orientation.

In view of Exercise 6.22 we can state an alternate definition of graded Gorenstein rings.

Definition 6.23. An artinian graded ring A is *Gorenstein* of socle degree d if and only if $A \cong D(A)(-d)$ as graded A-modules (degree preserving isomorphism).

6.3. **SLP for Gorenstein rings via Hessian matrices.** For this section let $R = \mathbb{F}[x_1,\dots,x_n]$ be a polynomial ring and R^* its graded dual. We will further assume that $\operatorname{char}(\mathbb{F}) = 0$.

In this section we use that R^* is isomorphic to $\mathbb{F}[X_1,\dots,X_n]$ with R-action $x_i \circ F = \frac{\partial F}{\partial X_i}$. We will use this description for R^*.

Lemma 6.24. *Let $F \in R^*_c$ and let $L = a_1x_1 + \cdots + a_nx_n \in R_1$. Then*

$$L^c \circ F = c! \cdot F(a_1,\dots,a_n).$$

Proof.

$$L^c \circ F = \sum_{i_1+\cdots+i_n=c} \frac{c!}{i_1!\cdots i_n!} a_1^{i_1}\cdots a_n^{i_n} x_1^{i_1}\cdots x_n^{i_n} \circ F = c!\cdot F(a_1,\ldots,a_n).$$

□

Definition 6.25 (Higher Hessians)**.** Let $F \in R^*$ be a homogeneous polynomial and let $B = \{b_1,\ldots,b_s\} \subseteq R_d$ be a finite set of homogeneous polynomials of degree $d \geq 0$. We call

$$\operatorname{Hess}^d_B(F) = [b_i b_j \circ F]_{1\leq i,j\leq s} \text{ and } \operatorname{hess}^d_B(F) = \det \operatorname{Hess}^d_B(F)$$

the d-th *Hessian matrix* and the d-th *Hessian determinant* of F with respect to B, respectively.

Remark 6.26. If $B = \{x_1,\ldots x_n\}$ then $\operatorname{Hess}^1_B(F) = [x_i x_j F]_{1\leq i,j\leq n} = \left[\frac{\partial F}{\partial X_i \partial X_j}\right]_{1\leq i,j\leq n}$ is the classical Hessian of F.

Hessians are useful in establishing the SLP for artinian Gorenstein rings.

Theorem 6.27 (Hessian criterion for SLP)**.** *Assume $\mathbb{F}$ is a field of characteristic zero. Let A be a graded artinian Gorenstein ring with Macaulay dual generator $F \in R^*_c$. Then A has the SLP if and only if*

$$\operatorname{hess}^i_{B_i}(F) \neq 0 \textit{ for } 0 \leq i \leq \lfloor \frac{c}{2} \rfloor$$

where B_i is some (any) basis of A_i.

Proof. From the hypothesis and Theorem 6.17 we have that $A = R/\operatorname{Ann}_R(F)$ has socle degree $d = \deg(F)$.

Since A is Gorenstein, A has symmetric Hilbert function, so A has SLP if and only if A has *SLP* in the narrow sense, i.e. there exists $L \in A_1$ such that for any $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$ the multiplication maps $L^{c-2i} : A_i \to A_{c-i}$ are vector space isomorphisms. Say $L = a_1x_1 + \cdots + a_nx_n$.

Recall that the isomorphism $A \cong D(A)(-c) = (R\circ F)(-c), a \mapsto a\circ F$ induces vector space isomorphisms $A_{d-i} \cong A_i^*$ also defined by $a \mapsto [b \mapsto b\circ(a\circ F) = (ba)\circ F]$. The composite map

$$T_i : A_i \xrightarrow{L^{c-2i}} A_{c-i} \xrightarrow{F} A_i^*$$

is an isomorphism if and only if multiplication by L^{c-2i} is an isomorphism. Let B_i be any basis for A_i and let B_i^* be its dual, which is a basis for A_i^*. The matrix $[t^{(i)}_{jk}]$for T_i with respect to these bases is defined as follows

$$T_i(b_j) = \sum_{k=1}^{s} t^{(i)}_{jk} b_k^*,$$

hence $t^{(i)}_{jk} = T_i(b_j)(b_k) = F(b_jL^{c-2i})(b_k) = (c-2i)!(b_jb_k\circ F)(a_1,\ldots,a_n)$, thus T_i is an isomorphism for some $L \in R_1$ if and only if

$$\operatorname{hess}^i_{B_i} F(a_1,\ldots,a_n) = \det\left[b_ib_j \circ F(a_1,\ldots,a_n)\right]_{1\leq i,j\leq s} \neq 0.$$

Overall the SLP holds if and only if for $0 \le i \le \lfloor \frac{c}{2} \rfloor$ the hessian determinant $\mathrm{hess}^i_{B_i} F$ does not vanish identically. $\square$

Example 6.28. Say $F = X^2 + Y^2 + Z^2$. Then with respect to the standard monomial basis for each R_i

$$\begin{aligned} \mathrm{hess}^0(F) &= F \\ \mathrm{hess}^1(F) &= \det \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 8 \\ \mathrm{hess}^i(F) &= 0 \text{ for } i \ge 2. \end{aligned}$$

Example 6.29. Let $G = XYW^3 + X^3ZW + Y^3Z^2$. Then $A = R/\operatorname{Ann}_R(G)$ has Hilbert function $1, 4, 10, 10, 4, 1$ and a basis for A_1 is $B_1 = \{x, y, z, w\}$ whereas a basis for A_2 is $B_2 = \{x^2, xy, xz, xw, y^2, yz, yw, z^2, zw, w^2\}$. Furthermore

$$\begin{aligned} \mathrm{hess}^0(G) &= G \\ \mathrm{hess}^1_{B_1}(G) &= \det \begin{pmatrix} 6XZW & W^3 & 3X^2W & 3X^2Z + 3YW^2 \\ W^3 & 6YZ^2 & 6Y^2Z & 3XW^2 \\ 3X^2W & 6Y^2Z & 2Y^3 & X^3 \\ 3X^2Z + 3YW^2 & 3XW^2 & X^3 & 6Z^2W \end{pmatrix} \neq 0 \\ \mathrm{hess}^2_{B_2}(G) &= \det \begin{pmatrix} 0 & 0 & 6\,w & 6\,z & 0 & 0 & 0 & 0 & 6\,x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6\,w \\ 6\,w & 0 & 0 & 6\,x & 0 & 0 & 0 & 0 & 0 & 0 \\ 6\,z & 0 & 6\,x & 0 & 0 & 0 & 6\,w & 0 & 0 & 6\,y \\ 0 & 0 & 0 & 0 & 0 & 12\,z & 0 & 12\,y & 0 & 0 \\ 0 & 0 & 0 & 0 & 12\,z & 12\,y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6\,w & 0 & 0 & 0 & 0 & 0 & 6\,x \\ 0 & 0 & 0 & 0 & 12\,y & 0 & 0 & 0 & 0 & 0 \\ 6\,x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6\,w & 0 & 6\,y & 0 & 0 & 6\,x & 0 & 0 & 0 \end{pmatrix} = 0. \end{aligned}$$

We conclude that the map $L : A_2 \to A_3$ fails to have maximum rank for all $L \in A_1$. However the map $L^3 : A_1 \to A_4$ does have maximum rank.

Exercise 6.30 (R. Gondim [9]). Let $x_1, \ldots, x_n$ and $u_1, \ldots, u_m$ be two sets of indeterminates with $n \ge m \ge 2$. Let $f_i \in \mathbb{F}[x_1, \ldots, x_n]_k$ and $g_i \in \mathbb{F}[u_1, \ldots, u_m]_e$ for $1 \le i \le s$ be linearly independent forms with $1 \le k < e$. If $s > \binom{m-1+k}{k}$, then

$$F = f_1 g_1 + \cdots + f_s g_s$$

is called a Perazzo form and $A = \mathbb{F}[x_1, \ldots, x_n, u_1, \ldots, u_m]$ is called a Perazzo algebra.

(1) Show that $\mathrm{hess}_k(F) = 0$ and so A does not have SLP.
(2) Make conjectures regarding the Hilbert functions of Perazzo algebras.
(3) Make conjectures regarding the WLP for Perazzo algebras.
(4) Do there exist two Perazzo algebras A and B having the same Hilbert function so that A has WLP and B does not?

Some answers to (2) and (3) can be found in [1].

Corollary 6.31. *Let $F \in \mathbb{F}[X_1, \dots, X_n], G \in \mathbb{F}[Y_1, \dots, Y_m]$ be homogeneous polynomials of the same degree. Then $A = \mathbb{F}[x_1, \dots, x_n]/\operatorname{Ann}_R(F)$ and $B = \mathbb{F}[y_1, \dots, y_m]/\operatorname{Ann}_R(G)$ have SLP if and only if*

$$C = \mathbb{F}[x_1, \dots, x_n, y_1, \dots, y_m]/\operatorname{Ann}_R(F+G) \text{ satisfies SLP.}$$

Proof. It turns out that for $1 \le i < \deg(F)$ a basis β of C_i is given by the union of a basis β' of A_i and a basis β'' of B_i (for a proof of this refer to Proposition 7.8 and Eq. (7.2)) and hence the hessians of $F+G$ look like

$$\begin{aligned}
\operatorname{Hess}^i(F+G) &= \left[b_i b_j(F+G)\right]_{b_i,b_j\in\beta} = \begin{bmatrix} b'_i b'_j(F) & 0 \\ 0 & b''_i b''_j(F) \end{bmatrix}_{b'_i,b'_j\in\beta', b''_i,b''_j\in\beta''} \\
&= \begin{bmatrix} \operatorname{Hess}^i(F) & 0 \\ 0 & \operatorname{Hess}^i(G) \end{bmatrix} \\
\operatorname{hess}^i(F+G) &= \operatorname{hess}^i(F)\operatorname{hess}^i(G).
\end{aligned}$$

Now we see that $\operatorname{hess}^i(F+G) \neq 0$ if and only if $\operatorname{hess}^i(F) \neq 0$ and $\operatorname{hess}^i(G) \neq 0$, which gives the desired conclusion. □

7. Topological ring constructions and the Lefschetz properties

We have seen in Section 2 that the Lefschetz properties emerged from algebraic topology. Now we return to this idea implementing some constructions that originate in topology at the ring level. The material in Section 7.1 is taken from [13] and the material in Section 7.2 is taken from [14].

7.1. **Fiber products and connected sums.**

We first consider the operation termed connected sum. A connected sum of manifolds along a disc is obtained by identifying a disk in each (with opposite orientations). One can more generally take connected sums by identifying two homeomorphic sub-manifolds, one from each summand. If the cohomology rings of the two summands are A and B and the cohomology ring of the common submanifold is T, then it turns out that the cohomology ring of the connected sum is $A\#_T B$, a ring that we term the connected sum of A and B over T in Definition 7.7.

To define a connected sum of rings we need a preliminary construction. Recall that an *oriented* AG algebra is a pair $(A, \int_A)$ with A an AG algebra and $\int_A$ an orientation as in Proposition 3.12. A choice of orientation on A also corresponds to a choice of Macaulay dual generator.

Exercise 7.1. Every orientation on A can be written as the function $\int_A : A \to K$ defined by $\int_A g = (g \circ F)(0)$ for some Macaulay dual generator F of A. The notation $(g \circ F)(0)$ refers to evaluating the element $g \circ F$ of R' at $X_1 = \cdots = X_n = 0$.

Next we discuss how the orientations of two AG algebras relate.

Definition 7.2 (Thom class)**.** Let $(A, \int_A)$ and $(T, \int_T)$ be two oriented AG K-algebras with of socle degree d for A and k for T, respectively, with $d \geq k$. Let $\pi : A \to T$ be a graded map. By [13, Lemma 2.1], there exists a unique homogeneous element $\tau_A \in A_{d-k}$ such that $\int_A(\tau_A a) = \int_T(\pi(a))$ for all $a \in A$; we call it the *Thom class* for $\pi : A \to T$.

Note that the Thom class for $\pi : A \to T$ depends not only on the map π, but also on the orientations chosen for A and T.

Example 7.3. Let $(A, \int_A)$ be an oriented AG K-algebra with socle degree $\operatorname{reg}(A) = d$. Consider $(K, \int_K)$ where $\int_K : K \to K$ is the identity map. Then the Thom class for the canonical projection $\pi : A \to K$ is the unique element $a_{soc} \in A_d$ such that $\int_A a_{soc} = 1$.

Exercise 7.4. Given a homomorphism $\pi : A \to T$ of AG algebras having dual generators F, H of degrees d and k, respectively, with $d \geq k$, show that the Thom class of Definition 7.2 is the unique element τ of A_{d-k} such that $\tau \circ F = H$.

Definition 7.5. Given graded $\mathbb{F}$-algebras A, B, and T, and graded $\mathbb{F}$-algebra maps $\pi_A\colon A \to T$ and $\pi_B\colon B \to T$, the *fiber product* of A and B over T (with respect to π_A and π_B) is the graded $\mathbb{F}$-subalgebra of $A \oplus B$

$$A \times_T B = \{(a, b) \in A \oplus B \mid \pi_A(a) = \pi_B(b)\}.$$

Let $\rho_1\colon A \times_T B \to A$ and $\rho_2\colon A \times_T B \to B$ be the natural projection maps. It is well known that fiber products are pullbacks in the category of $\mathbb{F}$ algebras and hence they satisfy the following universal property.

Lemma 7.6. *The fiber product $A \times_T B$ satisfies the following universal property: If C is another $\mathbb{F}$-algebra with maps $\phi_1\colon C \to A$ and $\phi_2\colon C \to B$ such that $\pi_A \circ \phi_1(c) = \pi_B \circ \phi_2(c)$ for all $c \in C$, then there is a unique $\mathbb{F}$-algebra homomorphism $\Phi\colon C \to A \times_T B$ which makes the diagram below commute:*

$$\begin{tikzcd} C \arrow[dr, dashed, "\Phi"] \arrow[drr, bend left, "\phi_1"] \arrow[ddr, bend right, "\phi_2"'] & & \\ & A \times_T B \arrow[r, "\rho_1"] \arrow[d, "\rho_2"'] & A \arrow[d, "\pi_A"] \\ & B \arrow[r, "\pi_B"'] & T. \end{tikzcd} \tag{7.1}$$

By [13, Lemma 3.7] the fiber product is characterized by the following exact sequence of vector spaces:

$$0 \to A \times_T B \to A \oplus B \xrightarrow{\pi_A - \pi_B} T \to 0, \tag{7.2}$$

whence the Hilbert function of the fiber product satisfies

$$H_{A \times_T B} = H_A + H_B - H_T. \tag{7.3}$$

Henceforth we assume that $\pi_A(\tau_A) = \pi_B(\tau_B)$, so that $(\tau_A, \tau_B) \in A \times_T B$.

Definition 7.7. The *connected sum* of the oriented AG K-algebras A and B over T is the quotient ring of the fiber product

$$A \times_T B := \{(a,b) \in A \oplus B \mid \pi_A(a) = \pi_B(b)\}$$

by the principal ideal generated by the pair of Thom classes (τ_A, τ_B), i.e.

$$A \#_T B = (A \times_T B)/\langle(\tau_A, \tau_B)\rangle.$$

By [13, Lemma 3.7] the connected sum is characterized by the following exact sequence of vector spaces:

$$0 \to T(k-d) \to A \times_T B \to A \#_T B \to 0. \tag{7.4}$$

Therefore, the Hilbert series of the connected sum satisfies

$$HF_{A\#_T B}(t) = HF_A(t) + HF_B(t) - (1 + t^{d-k})HF_T(t). \tag{7.5}$$

When $T = \mathbb{F}$ we have an easy description of the fiber product and connected sum.

Proposition 7.8. *Let $R = \mathbb{F}[x_1, \dots, x_n]$, $R' = \mathbb{F}[y_1, \dots, y_m]$ be polynomial rings. Let $\left(A = R/I, \int_A\right)$ and $\left(B = R'/I', \int_B\right)$ be oriented AG algebras each with socle degree d, and let $\pi_A \colon A \to \mathbb{F}$ and $\pi_B \colon B \to \mathbb{F}$ be the natural projection maps with Thom classes $\tau_A \in A_d$ and $\tau_B \in B_d$. Then the fiber product $A \times_{\mathbb{F}} B$ has a presentation*

$$A \times_{\mathbb{F}} B \cong \frac{\mathbb{F}[x_1, \dots, x_n, y_1, \dots, y_m]}{(x_i y_j \mid 1 \le i \le n, 1 \le j \le m) + I + I'}.$$

and the connected sum $A \#_{\mathbb{F}} B$ has a presentation

$$A \#_{\mathbb{F}} B \cong \frac{\mathbb{F}[x_1, \dots, x_n, y_1, \dots, y_m]}{(x_i y_j \mid 1 \le i \le n, 1 \le j \le m) + I + I' + (\tau_A + \tau_B)}.$$

In particular, if A and B are standard graded then so are $A \times_{\mathbb{F}} B$ and $A \#_{\mathbb{F}} B$.

Example 7.9 (Standard graded fiber product and connected sum)**.** Let $A = \mathbb{F}[x,y]/(x^2, y^4)$ and $B = \mathbb{F}[u,v]/(u^3, v^3)$ each with the standard grading $\deg(x) = \deg(y) = \deg(u) = \deg(v) = 1$. Let $T = \mathbb{F}[z]/(z^2)$, and define maps $\pi_A : A \to T$, $\pi_A(x) = z$, $\pi_A(y) = 0$ and $\pi_B \colon B \to T$, $\pi_B(u) = z$, $\pi_B(v) = 0$. Then the fiber product $A \times_T B$ is generated as an algebra by elements $z_1 = (y, 0)$, $z_2 = (x, u)$, and $z_3 = (0, v)$, all having degree one. One can check that it has the following presentation:

$$A \times_T B = \frac{\mathbb{F}[z_1, z_2, z_3]}{\langle z_1^4, z_2^3, z_3^3, z_1 z_3, z_1 z_2^2\rangle}. \tag{7.6}$$

The Hilbert function of the fiber product is

$$\begin{aligned} H(A \times_T B) =& (1,3,5,4,2) \\ =& (1,2,2,2,1) + (1,2,3,2,1) - (1,1,0,0,0) \\ =& H(A) + H(B) - H(T). \end{aligned}$$

Fix orientations on A, B, and T by $\int_A \colon xy^3 \mapsto 1$, $\int_B \colon u^2v^2 \mapsto 1$, and $\int_T \colon z \mapsto 1$, respectively. Then the Thom classes for $\pi_A \colon A \to T$ and $\pi_B \colon B \to T$ are, respectively,

$\tau_A = y^3$, $\tau_B = uv^2$. Note that $\pi_A(\tau_A) = 0 = \pi_B(\tau_B)$, hence $(\tau_A, \tau_B) \in A \times_T B$, and in terms of Presentation (7.6) we have $(\tau_A, \tau_B) = z_1^3 + z_2 z_3^2$. Therefore we see that

$$A\#_T B = \frac{\mathbb{F}[z_1, z_2, z_3]}{\langle z_1^4, z_2^3, z_3^3, z_1 z_3, z_1 z_2^2, z_1^3 + z_2 z_3^2 \rangle}. \tag{7.7}$$

The Hilbert function of the connected sum is

$$\begin{aligned} H(A\#_T B) =& (1,3,5,3,1) \\ =& (1,2,2,2,1) + (1,2,3,2,1) - (1,1,0,0,0) - (0,0,0,1,1) \\ =& H(A) + H(B) - H(T) - H(T)[3] \end{aligned}$$

However, if $T \neq \mathbb{F}$ the presentation of the connected sum and fiber product can be complicated and they need not be standard graded.

Example 7.10 (Non-standard graded fiber product and connected sum)**.** Let

$$A = \mathbb{F}[x]/(x^4),\ B = \mathbb{F}[u,v]/(u^3, v^2),\ T = \mathbb{F}[z]/(z^2),$$

have Hilbert functions $H(A) = (1,1,1,1)$ and $H(B) = (1,2,2,1)$. Define maps $\pi_A \colon A \to T$, $\pi_A(x) = z$ and $\pi_B \colon B \to T$, $\pi_B(u) = z$, $\pi_B(v) = 0$. Then the fibered product has the presentation

$$A \times_T B = \frac{\mathbb{F}[z_1, z_2, z_3]}{(z_1^4, z_2^2, z_3^2, z_1 z_3, z_1^2 z_2 - z_2 z_3)}, \quad \text{where} \begin{cases} z_1 = & (x, u) \\ z_2 = & (0, v) \\ z_3 = & (0, u^2). \end{cases}$$

Here z_1, z_2 have degree one, and z_3 has degree two. We then have a presentation for the connected sum $C = A\#_T B = A \times_T B/(\tau)$, whence

$$A\#_T B \cong \frac{\mathbb{F}[z_1, z_2, z_3]}{(z_1^4, z_2^2, z_3^2, z_1 z_3, z_1^2 z_2 - z_2 z_3, (z_1^2 - z_3) + z_1 z_2)} \cong \frac{\mathbb{F}[z_1, z_2]}{(z_1^3 + z_1^2 z_2, z_2^2)}.$$

It has Hilbert function $H(C) = (1,2,2,1) = H(A) + H(B) - H(T) - H(T)[1]$ as in (7.5). It is interesting to note that the connected sum $A\#_T B$ has a standard grading whereas the fibered product $A \times_T B$ does not.

Finally, we have the following result which shows how the Lefschetz properties of the components influence the Lefschetz property of the fiber product and connected sum.

Theorem 7.11. *(1) If A and B are AG algebras of the same socle degree that each have the SLP, then the fiber product $D = A \times_{\mathbb{F}} B$ over a field $\mathbb{F}$ also has the SLP. If A and B have the standard grading, then the converse holds as well.*

(2) If A and B both have the SLP, then the connected sum $C = A\#_{\mathbb{F}} B$ over a field $\mathbb{F}$ also has the SLP. If A and B have the standard grading, then the converse holds as well.

(3) Let A, T be AG algebras with socle degrees d, k respectively and let $\pi_A \colon A \to T$ be a surjective ring homomorphism such that its Thom class τ_A satisfies $\pi_A(\tau_A) = 0$. Let x be an indeterminate of degree one, set $B = T[x]/(x^{d-k+1})$, and define $\pi_B \colon B \to T$ to be the natural projection map satisfying $\pi_B(t) = t$ and $\pi_B(x) = 0$. In this setup, if A and T both satisfy the SLP, then the fiber

product $A \times_T B$ also satisfies the SLP. Moreover if the field $\mathbb{F}$ is algebraically closed, then the connected sum $A\#_T B$ also satisfies the SLP.

(4) Let A and B be standard graded AG algebras of socle degree d satisfying the SLP, and let T be a graded AG algebra of socle degree k, with $k < \lfloor \frac{d-1}{2} \rfloor$, endowed with surjective $\mathbb{F}$-algebra homomorphisms $\pi_A : A \to T$ and $\pi_B : B \to T$. Then the resulting fiber product $A \times_T B$ and the connected sum $A\#_T B$ both satisfy the WLP.

Example 7.12. Take

$$F = xy(xz - yt) \in K[x, y, z, t]$$

and set $A = K[x, y, z, t]/\operatorname{Ann}(F)$. Then

$$\operatorname{Ann}(F) = (zt, xz + yt, x^2t, y^2z, x^2y^2, x^3, y^3, z^2, t^2),$$

A is a connected sum

$$A = K[x, y, z]/\operatorname{Ann}(x^2yz)\#_{K[x,y]/\operatorname{Ann}(xy)}K[x, y, t]/\operatorname{Ann}(xy^2t),$$

and the Hilbert function of A is $(1, 4, 6, 4, 1)$. By Theorem 7.11 (4), since the summands of A are monomial complete intersections, A has WLP if the characteristic of $\mathbb{F}$ is 0.

Example 7.13. Take

$$F = x^3yz - xy^3t = xy(x^2z - y^2t) \in K[x, y, z, t]$$

and set $A = K[x, y, z, t]/\operatorname{Ann}(F)$. Then

$$\operatorname{Ann}(F) = (z^2, t^2, tz, x^2t, y^2z, x^2z + y^2t, y^4, x^2y^2, x^4),$$

A is a connected sum

$$A = K[x, y, z]/\operatorname{Ann}(x^3yz)\#_{K[x,y]/\operatorname{Ann}(xy)}K[x, y, t]/\operatorname{Ann}(xy^3t),$$

and the Hilbert function of A is $(1, 4, 7, 7, 4, 1)$.

The Hessian matrix of F of order two is of the following form

$$\operatorname{Hess}^2(F) = 6 \begin{pmatrix} 0 & y & x & z & 0 & 0 & 0 \\ y & 0 & 0 & x & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 0 & 0 \\ z & x & 0 & 0 & 0 & -y & -t \\ 0 & 0 & 0 & 0 & 0 & 0 & -y \\ 0 & 0 & 0 & -y & 0 & 0 & -x \\ 0 & 0 & 0 & -t & -y & -x & 0 \end{pmatrix}$$

and it has vanishing determinant. According to the Hessian criteria Theorem 6.27 A does not have WLP because in this case the second Hessian corresponds to the multiplication map from degree 2 to degree 3. Note that the socle degrees don't satisfy the condition in Theorem 7.11 since $2 = k = \lfloor \frac{d-1}{2} \rfloor = \frac{5-1}{2}$.

7.2. **Cohomological blowups.** The second construction is inspired by the geometric operation of blowing up a smooth projective algebraic variety. The blow-up of such a space at a point replaces the point with the set of all directions through the point, that is, a projective space. More generally one can blow up a subset and replace it with another space called am *exceptional divisor*. The cohomology ring of the blow-up can be determined based on the cohomology ring of the original variety (called A below), that of the subvariety being blown up (called T below) and the way the latter sits inside the former, specifically captured via the cohomology class of the *normal bundle* of the subvariety, encoded via a polynomial $f_A(\xi)$ below.

We now explain the algebraic construction for the cohomology ring of a blowup.

Definition 7.14 (Cohomological Blow-Up)**.** For oriented AG algebras A and T of socle degrees $d > k$, respectively, and surjective degree-preserving algebra map $\pi\colon A \to T$ with Thom class $\tau \in A_n$ where $n = d - k$, set $K = \operatorname{Ker}(\pi)$. Given a homogeneous monic polynomial $f_A(\xi) = \xi^n + a_1\xi^{n-1} + \cdots + a_n \in A[\xi]$ of degree n with homogeneous elements $a_i \in A_i$ for $1 \le i \le n$ and with $a_n = \lambda \cdot \tau$ for some non-zero constant λ, we call the AG algebra $\tilde{A}$ below a *cohomological blow up of A along* π or BUG for short

$$\tilde{A} = \frac{A[\xi]}{(\xi \cdot K, \underbrace{\xi^n + a_1\xi^{n-1} + \cdots + \lambda \cdot \tau}_{f_A(\xi)})}.$$

Setting $t_i = \pi(a_i)$ for $1 \le i \le n-1$, the AG algebra

$$\tilde{T} = \frac{T[\xi]}{\underbrace{(\xi^n + t_1\xi^{n-1} + \cdots + \lambda \cdot \pi(\tau))}_{f_T(\xi)}}$$

is called the *exceptional divisor of T with parameters* $(t_1, \ldots, t_{n-1}, \lambda)$. These algebras fit in the following commutative diagram, where we refer to A as the *cohomological blow down of $\tilde{A}$ along $\hat{\pi}$*.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\pi"'] & \tilde{A} \arrow[d, "\hat{\pi}"] \\ T \arrow[r] & \tilde{T}. \end{tikzcd}$$

Since $\tilde{T}$ is a quotient of a 1-dimensional Gorenstein ring by a non zero-divisor, it is clear that $\tilde{T}$ is AG. It is shown in [14] that the condition that the last term of $f_A(\xi)$ be a scalar multiple of the Thom class τ is precisely equivalent to $\tilde{A}$ being AG.

Example 7.15. Let

$$A = \frac{\mathbb{F}[x,y]}{(x^3, y^3)} \xrightarrow{\pi} T = \frac{\mathbb{F}[x,y]}{(x^2, y)}$$

where $\pi(x) = x$ and $\pi(y) = 0$. Note $K = \operatorname{Ker}(\pi) = (x^2, y)$. Orient A and T with socle generators $a_{soc} = x^2y^2$ and $t_{soc} = x$; then the Thom class of π is $\tau = xy^2 \in A_3$. Set

$f_T(\xi) = \xi^3 + x\xi^2 \in T[\xi]$ and let $\tilde{T}$ be the associated exceptional divisor algebra:

$$\tilde{T} = \frac{T[\xi]}{(f_T(\xi))} = \frac{\mathbb{F}[x, y, \xi]}{(x^2, y, \xi^3 + x\xi^2)}.$$

Consider $f_A(\xi) = \xi^3 + x\xi^2 + xy^2 \in A[\xi]$. This gives rise to the BUG

$$\tilde{A} = \frac{A[\xi]}{(\xi \cdot K, f_A(\xi))} = \frac{\mathbb{F}[x, y, \xi]}{(x^3, y^3, x^2\xi, y\xi, \xi^3 + x\xi^2 + xy^2)}$$

which has basis

$$\left\{1, x, y, \xi, x^2, xy, y^2, x\xi, \xi^2, x^2y, xy^2, x\xi^2, x^2y^2\right\}$$

and Hilbert function $H(\tilde{A}) = (1, 3, 5, 3, 1)$. Here the socle of $\tilde{A}$ is generated by $\tilde{a}_{soc} = a_{soc} = x^2y^2$, hence $\tilde{A}$ is Gorenstein, as expected.

We are now ready to discuss the Lefschetz properties for cohomological blow-up algebras.

Theorem 7.16. *Let $\mathbb{F}$ be an infinite field and let $\pi : A \to T$ be a surjective homomorphism of graded AG $\mathbb{F}$-algebras of socle degrees $d > k$ respectively such that both A and T have SLP. Assume that characteristic $\mathbb{F}$ is zero or characteristic F is $p > d$. Then every cohomological blow-up algebra of A along T satisfies SLP.*

The following example shows that the converse of Theorem 7.16 is not true: if the cohomological blowup $\tilde{A}$ has SLP it does not follow that A has SLP. In other words, while the process of blowing up preserves SLP, the process of blowing down does not preserve SLP, nor even WLP.

Example 7.17. As in Exercise C.3, the following example, originally due to U. Perazzo, but re-examined more recently by R. Gondim and F. Russo [10], is an AG algebra with unimodal Hilbert function which does not have SLP or WLP:

$$\begin{aligned} A &= \frac{\mathbb{F}[x, y, z, u, v]}{\mathrm{Ann}(XU^2 + YUV + ZV^2)} \\ &= \frac{\mathbb{F}[x, y, z, u, v]}{(x^2, xy, y^2, xz, yz, z^2, u^3, u^2v, uv^2, v^3, xv, zu, xu - yv, zv - yu)}. \end{aligned}$$

Taking the quotient T of A given by the Thom class $\tau = u^2$ yields

$$T = \frac{\mathbb{F}[x, y, z, u, v]}{\mathrm{Ann}(X)} = \frac{\mathbb{F}[x, y, z, u, v]}{(x^2, y, z, u, v)} \cong \frac{\mathbb{F}[x]}{(x^2)}.$$

Fix a parameter $\lambda \in \mathbb{F}$ and define polynomials $f_T(\xi) \in T[\xi]$ and $f_A(\xi) \in A[\xi]$ by

$$f_T(\xi) = \xi^2 - \lambda x\xi \quad \text{and } f_A(\xi) = \xi^2 - \lambda x\xi + u^2.$$

Denoting the ideal of relations of A by I we obtain the cohomological blowup

$$\tilde{A} = \frac{\mathbb{F}[x, y, z, u, v, \xi]}{I + \xi(y, z, u, v) + (f_A(\xi))},$$

which has Hilbert function $H(\tilde{A}) = H(A) + H(T)[1] = (1, 6, 6, 1)$. Fix $\mathbb{F}$-bases

$$\tilde{A}_1 = \mathrm{span}_{\mathbb{F}} \{x, y, z, u, v, \xi\}, \quad \text{and} \quad \tilde{A}_2 = \mathrm{span}_{\mathbb{F}} \left\{u^2, uv, v^2, yv, yu, -x\xi\right\}$$

and let $\ell \in \tilde{A}_1$ be a general linear form

$$\ell = ax + by + cz + du + ev + f\xi.$$

Then the matrix for the Lefschetz map $\times\ell\colon \tilde{A}_1 \to \tilde{A}_2$ and its determinant are given by

$$M = \begin{pmatrix} 0 & 0 & 0 & d & 0 & -f \\ 0 & 0 & 0 & e & d & 0 \\ 0 & 0 & 0 & 0 & e & 0 \\ d & e & 0 & a & b & 0 \\ 0 & d & e & b & c & 0 \\ -f & 0 & 0 & 0 & 0 & -(a+\lambda f) \end{pmatrix} \Rightarrow \det(M) = f^2e^4.$$

Thus ℓ is a strong Lefschetz element for $\tilde{A}$ if and only if $e \cdot f \neq 0$. In particular $\tilde{A}$ satisfies SLP and also WLP.

Surprisingly, the analogous result to Theorem 7.16 does not hold for the weak Lefschetz property. We now give an example illustrating that blowing up does not preserve WLP.

Exercise 7.18. Consider the following algebra

$$\begin{aligned} A &= \frac{\mathbb{F}[x,y,z,u,v]}{\operatorname{Ann}(XU^6 + YU^4V^2 + ZU^5V)} \\ &= \frac{\mathbb{F}[x,y,z,u,v]}{(yz, xz, xy, vy - uz, vx, ux - vz, u^5y, u^5v^2, u^6v, u^7, v^3, x^2, y^2, z^2)} \end{aligned}$$

and its quotient corresponding to the Thom class $\tau = u^3$

$$\begin{aligned} T &= \frac{\mathbb{F}[x,y,z,u,v]}{\operatorname{Ann}(XU^3 + YUV^2 + ZU^2V)} \\ &= \frac{\mathbb{F}[x,y,z,u,v]}{(z^2, yz, xz, y^2, xy, vy - uz, x^2, vx, ux - vz, u^2y, v^3, u^2v^2, u^3v, u^4)} \end{aligned}$$

Consider also the cohomological blowup

$$\tilde{A} = \frac{\mathbb{F}[x,y,z,u,v,\xi]}{I + \xi \cdot K + (\xi^3 - u^3)}.$$

(a) Compute the Hilbert functions of A and T respectively.
(b) Show that both A and T satisfy WLP, but not SLP.
(c) Show that the BUG $\tilde{A}$ does not satisfy WLP.

In Exercise 7.18 the Thom class of the map $A \to T$ has degree 3. This is the minimal possible value for such an example based on the following result.

Remark 7.19. Let $\mathbb{F}$ be an infinite field and let $\pi : A \to T$ be a surjective homomorphism of graded AG $\mathbb{F}$-algebras such that the difference between the socle degrees of A and T is at most 2 and A and T both satisfy WLP. Then every cohomological blow-up algebra of A along π satisfies WLP.

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Appendix A. Exercise session 1: Computations in Macaulay2

Use Macaulay2 to solve the exercises in this section!

Exercise A.1. Determine whether the algebra
$$\frac{\mathbb{Q}[x,y,z]}{(x^2+y^2+z^2,\ xyz,\ z^4-3xz^3)}$$
is artinian.

Exercise A.2. Compute the Hilbert series of the algebra
$$A=\frac{\mathbb{Z}/3\mathbb{Z}[x,y,z]}{(x^{10},y^{10},z^{10},x^3y^3z^3)}.$$
Is $x+y+z$ a weak Lefschetz element of A?

Exercise A.3. Let $R=\mathbb{Q}[x_1,x_2,x_3,x_4]$ and $\mathfrak{m}=(x_1,x_2,x_3,x_4)$. Does the algebra $R/(\mathfrak{m}^5+x_1\mathfrak{m}^2+(x_2^3))$ have WLP?

Exercise A.4. Build a function in Macaulay2 that takes as input an artinian standard graded algebra A and an element $\ell\in A_1$ and returns `true` of `false` given by whether ℓ is a weak Lefschetz element of A.

Hint: use the result of Exercise 4.3 (also stated as Exercise B.2 below).

Exercise A.5. Use your function from Exercise A.4 to explore the WLP of the algebra
$$\frac{\mathbb{Z}/p\mathbb{Z}[x_1,\dots,x_n]}{(x_1^n,\dots,x_n^n,\ x_1\cdots x_{n-1}(x_1+x_n))}$$
for some integer $n\geq 3$, and some prime number p. For which n and p can you detect WLP?

Results related to Exercise A.2 and Exercise A.5 can be found in [19].

Appendix B. Exercise session 2: Lefschetz Properties

A $(*)$ *denotes that at least some portion of the exercise is an open (research) question.*

Exercise B.1. The rings $A=\mathbb{F}[x_1,\dots,x_n]/(x_1^{d_1},\dots,x_n^{d_n})$, where $d_1,\dots,d_n\geq 1$ are integers, are called monomial complete intersections.

(1) Prove that monomial complete intersections are the only complete intersection rings of the form $\mathbb{F}[x_1,\dots,x_n]/I$ where I is an ideal generated by monomials.
(2) Prove that monomial complete intersections are the only artinian Gorenstein rings of the form $\mathbb{F}[x_1,\dots,x_n]/I$ where I is an ideal generated by monomials.

Exercise B.2 (Equivalent WLP statements)**.** Prove that for an artinian graded $\mathbb{F}$-algebra A the following are equivalent:

(1) $L\in A_1$ is a weak Lefschetz element for A.
(2) For all $0\leq i\leq c-1$ the map $\times L: A_i\to A_{i+1}$ has rank $\min\{h_A(i),h_A(i+1)\}$.
(3) For all $0\leq i\leq c-1$ $\dim_{\mathbb{F}}([(L)]_{i+1})=\min\{h_A(i),h_A(i+1)\}$.
(4) For all $0\leq i\leq c-1$ $\dim_{\mathbb{F}}([A/(L)]_{i+1})=\max\{0,h_A(i+1)-h_A(i)\}$.

(5) For all $0 \le i \le c-1$ $\dim_{\mathbb{F}}([0 :_A L]_i) = \max\{0, h_A(i) - h_A(i+1)\}$.

Exercise B.3. Let $\mathbb{F}$ be a field of characteristic zero and let
$$A = \frac{\mathbb{F}[x,y,z]}{(x^3, y^3, z^3, (x+y+z)^3)}.$$

(1) Find the Hilbert function of A.
(2) Prove that A satisfies WLP but not SLP.

Exercise B.4. $(*)$ With help from a computer make conjectures regarding the WLP and SLP for monomial complete intersections in positive characteristics. A characterization is known for SLP, but not for WLP. See [4, 20] for related work.

Exercise B.5. Let $\mathbb{F}[x,y]_d$ be the vector space of polynomials of degree d in $\mathbb{F}[x,y]$. Prove:

(1) $E = x\frac{\partial}{\partial y}, H = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, F = y\frac{\partial}{\partial x}$ form an $\mathfrak{sl}_2$-triple.
(2) Prove that the monomial $x^a y^b$ is an eigenvector of H with eigenvalue $a-b \in \mathbb{Z}$. In particular the eigenvalues of H on $\mathbb{F}[x,y]_d$ are $d, d-2, d-4, \ldots, 4-d, 2-d, -d$.
(3) Prove that a basis of $\mathbb{F}[x,y]_d$ is $y^d, E(y^d), E^2(y^d), \ldots, E^d(y^d)$.
(4) Find a basis that satisfies the properties given by Theorem 5.10.

Pictorially this can be summarized as

$$\begin{tikzcd}
0 & \mathbb{F}y^d \arrow[l, "F"'] \arrow[r, bend right, "E"'] & \mathbb{F}xy^{d-1} \arrow[l, bend right, "F"'] \arrow[r, bend right, "E"'] & \cdots \arrow[l, bend right, "F"'] \arrow[r, bend right, "E"'] & \mathbb{F}x^{d-1}y \arrow[l, bend right, "F"'] \arrow[r, bend right, "E"'] & \mathbb{F}x^d \arrow[l, bend right, "F"'] \arrow[r, "E"'] & 0
\end{tikzcd}$$

Exercise B.6. (1) Suppose that V is a representation of $\mathfrak{sl}_2$ and that the eigenvalues of H on V are $2, 1, 1, 0, -1, -1, -2$. Show that the irreducible decomposition of V is $V \cong \mathbb{F}[x,y]_2 \oplus \mathbb{F}[x,y]_1 \oplus \mathbb{F}[x,y]_1$.
(2) Prove that if V is any representation of $\mathfrak{sl}_2$ then its irreducible decomposition is determined by the eigenvalues of H.

Exercise B.7. Let V be an $\mathfrak{sl}_2$ representation and set $W_k = \{v \in V \mid H(v) = kv\}$.

(1) Show that $\dim_{\mathbb{F}} W_k = \dim_{\mathbb{F}} W_{-k}$.
(2) Prove that $E^k : W_{-k} \to W_k$ is an isomorphism.
(3) Show that $\dim_{\mathbb{F}} W_{k+2} \le \dim_{\mathbb{F}} W_k$ for all $k \ge 0$, that is, the two sequences
$$\ldots, \dim_{\mathbb{F}} W_4, \dim_{\mathbb{F}} W_2, \dim_{\mathbb{F}} W_0, \dim_{\mathbb{F}} W_{-2}, \dim_{\mathbb{F}} W_{-4}, \ldots$$
$$\ldots, \dim_{\mathbb{F}} W_3, \dim_{\mathbb{F}} W_1, \dim_{\mathbb{F}} W_{-1}, \dim_{\mathbb{F}} W_{-3}, \ldots$$
are unimodal.

Appendix C. Exercise session 3: Gorenstein rings

A $(*)$ denotes that at least some portion of the exercise is an open (research) question.

Exercise C.1. Generalize Example 6.14 to find the inverse system of the ideal defining a point in projective n-space and the inverse systems of all of the powers of this ideal.

Exercise C.2. For any homogeneous polynomial $F \in R^*$ of degree d, prove

$$\mathrm{Ann}_R(F)^{\perp} := \{g \in R^* \mid f \bullet g = 0, \forall f \in \mathrm{Ann}_R(F)\}$$

is equal to $R \bullet F$. This is an instance of Macaulay's double annihilator theorem.

Hint: start by showing the equality is true in degree d, then use the R-module structure.

Exercise C.3 (R. Gondim [9])**.** Let $x_1, \dots, x_n$ and $u_1, \dots, u_m$ be two sets of indeterminates with $n \geq m \geq 2$. Let $f_i \in \mathbb{F}[x_1, \dots, x_n]_k$ and $g_i \in \mathbb{F}[u_1, \dots, u_m]_e$ for $1 \leq i \leq s$ be linearly independent forms with $1 \leq k < e$. If $s > \binom{m-1+k}{k}$, then

$$F = f_1 g_1 + \cdots + f_s g_s$$

is called a Perazzo form and $A = \mathbb{F}[x_1, \dots, x_n, u_1, \dots, u_m]$ is called a Perazzo algebra.

(1) Show that $\mathrm{hess}_k(F) = 0$ and so A does not have SLP.
(2) Make conjectures regarding the Hilbert functions of Perazzo algebras.
(3) Make conjectures regarding the WLP for Perazzo algebras.
(4*) Do there exist two Perazzo algebras A and B having the same Hilbert function so that A has WLP and B does not?

Some answers to (2) and (3) can be found in [1].

Exercise C.4. Show that every orientation on an AG algebra A can be written as the function $\int_A : A \to K$ defined by $\int_A g = (g \circ F)(0)$ for some Macaulay dual generator F of A, where $(g \circ F)(0)$ refers to evaluating the element $g \circ F$ of R' at $X_1 = \cdots = X_n = 0$.

Exercise C.5. Show that there exists a surjective homomorphism $\pi : A \to T$ of AG algebras having dual generators F, H of degrees $d \geq k$ if and only if there exists $\tau \in A_{d-k}$ such that $\tau \circ F = H$. *Hint:* You may use the Thom class in Definition 7.2.

Exercise C.6. Consider the following algebra

$$\begin{aligned} A &= \frac{\mathbb{F}[x,y,z,u,v]}{\mathrm{Ann}(XU^6 + YU^4V^2 + ZU^5V)} \\ &= \frac{\mathbb{F}[x,y,z,u,v]}{(yz, xz, xy, vy - uz, vx, ux - vz, u^5y, u^5v^2, u^6v, u^7, v^3, x^2, y^2, z^2)} \end{aligned}$$

and its quotient corresponding to the Thom class $\tau = u^3$

$$\begin{aligned} T &= \frac{\mathbb{F}[x,y,z,u,v]}{\mathrm{Ann}(XU^3 + YUV^2 + ZU^2V)} \\ &= \frac{\mathbb{F}[x,y,z,u,v]}{(z^2, yz, xz, y^2, xy, vy - uz, x^2, vx, ux - vz, u^2y, v^3, u^2v^2, u^3v, u^4)} \end{aligned}$$

Let K be the kernel of the canonical surjection $\pi : A \to T$. Consider the cohomological blow up

$$\tilde{A} = \frac{\mathbb{F}[x,y,z,u,v,\xi]}{I + \xi \cdot K + (\xi^3 - u^3)}.$$

(a) Compute the Hilbert functions of A and T respectively. Feel free to use *Macaulay2*.
(b) Show that both A and T satisfy WLP, but not SLP. You may use *Macaulay2*.
(c) Show that the BUG $\tilde{A}$ does not satisfy WLP.