Math 918 - Fall 2019

The Lefschetz properties

Contents

Chapter 1

What is this course about?

Lecture 1 – August 27, 2019

I will give an introduction to the origins of the algebraic Lefschetz properties. The motivation for this topic comes from algebraic topology, so we will spend just a bit of time talking about how the Lefschetz property arises there. This material will feel more advanced than the next lectures.

Cohomology rings and the Hard Lefschetz theorem

Let R be a commutative ring and let X be a topological space (such as projective space $\mathbb{P}^n$ or the n-dimensional sphere S^n). Let's recall the notion of *cohomology* of X with coefficients in R.

First, one can think of X as being made out of simple cells (or at least one can approximate X in this manner). This endows X with a *cell complex (CW-complex)* structure.

Example 1.1 (CW structure on sphere)**.** For example, the 2-dimensional sphere S^2 can be obtained from taking a point (0-dimensional cell) and glueing a 2-dimensional disc onto it along its entire boundary. So the CW-structure of S^2 is

$$S^2 = \text{pt} + \text{2-dimensional disc}$$

and more generally

$$S^n = \text{pt} + \text{n-dimensional disc.}$$

Example 1.2 (CW structure on real projective space)**.** Consider first $\mathbb{P}^n_{\mathbb{R}}$. It can be written as $S^n/\{\pm 1\}$. If we take a CW structure on S^n with two cells in each dimension, with the -1-action swaps the cells, thus they become identified in the quotient and so $\mathbb{P}^n_{\mathbb{R}}$ has a CW structure with one cell in each dimension.

$$\mathbb{P}^n_{\mathbb{R}} = \text{pt} + \text{1-dimensional cell} + \cdots + \text{n-dimensional cell.}$$

Next consider $\mathbb{P}^n_{\mathbb{C}}$. This has a cell in every even dimension:

$$\mathbb{P}^n_{\mathbb{C}} = \text{pt} + \text{2-dimensional cell} + \cdots + \text{2n-dimensional cell.}$$

Proceeding towards homology, we define a *chain complex* $\mathbf{C}_\bullet(X)$ by letting $C_n(X)$ be the free R-module generated by the n-dimensional cells of X. There are so-called boundary maps which fit into the following sequence

$$\mathbf{C}_\bullet(X) : 0 \leftarrow R^{\#0\text{-cells}} \leftarrow R^{\#1\text{-cells}} \leftarrow \cdots \leftarrow R^{\#\dim(X)\text{-cells}} \leftarrow 0.$$

There is also a dual version called the *cochain complex* of X with coefficients in R

$$\mathbf{C}^\bullet(X) = \operatorname{Hom}(\mathbf{C}_\bullet(X), R) : 0 \to R^{\#0\text{-cells}} \xrightarrow{\partial_1} R^{\#1\text{-cells}} \xrightarrow{\partial_2} \cdots \xrightarrow{\partial_n} R^{\#\dim(X)\text{-cells}} \to 0.$$

Definition 1.3. The *cohomology groups* of X are defined as

$$H^i(X, R) = H^i\left(\mathbf{C}^\bullet(X)\right) = \operatorname{Ker} \partial_i / \operatorname{Im} \partial_{i-1}.$$

Example 1.4. Based on the previous examples we have

$$\mathbf{C}^\bullet(S^n) : 0 \to R \to 0 \to 0 \to \ldots \to R \to 0$$

$$H^i(S^n, R) = \begin{cases} R & i = 0, n \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{C}^\bullet(\mathbb{P}^n_{\mathbb{C}}) : 0 \to R \to 0 \to R \to 0 \to R \to \ldots \to R \to 0$$

$$H^i(\mathbb{P}^n_{\mathbb{C}}, R) = \begin{cases} R & i = \text{even} \\ 0 & i = \text{odd} \end{cases}$$

The special property of these cohomology groups that allows us to do even more algebra is that they can be assembled into a graded ring.

Definition 1.5. The *cohomology ring* of X is

$$H^\bullet(X, R) = \bigoplus_{i \geq 0} H^i(X, R).$$

To explain a multiplication structure on this ring we need to define a map called the *cup product*

$$H^m(X, R) \times H^n(X, R) \to H^{m+n}(X, R).$$

For this recall the Künneth isomorphism: for two topological spaces X and Y if one of X or Y has R-torsion free homology (true when R is a field) and has finitely many cells in each dimensions, there is an isomorphism $k : H^\bullet(X \times Y, R) \cong H^\bullet(X, R) \otimes_R H^\bullet(Y, R)$. The composite with the diagonal map

$$H^\bullet(X, R) \otimes_R H^\bullet(X, R) \xrightarrow{\cong} H^\bullet(X \times X, R) \xrightarrow{\Delta^*} H^\bullet(X, R)$$

defines the cup product by $x \cup y = \Delta^* k(x \otimes y)$. The cup product is not commutative on the nose, but it is what we call *graded commutative*: if $x \in H^m(X, R)$ set $|x| = m$ to be the degree of x. Then

$$x \cup y = (-1)^{|x||y|} x \cup y. \tag{1.0.1}$$

Example 1.6 (Homology ring of spheres)**.** From Example 1.4 we have $H^\bullet(S^n, R) = R \oplus R$. The unit of this ring is $1 = (1, 0)$ and we set $\epsilon = (0, 1) \in H^n(S^n, R)$. Then $\epsilon^2 = \epsilon \cup \epsilon \in H^{2n}(S^n, R) = 0$, so

$$H^\bullet(S^n, R) = R[\epsilon]/(e^2) \text{ with } |e| = n.$$

Suppose $R = \mathbb{F}$ is a field. Applying the Künneth formula to the torus $T^n = S^1 \times \cdots \times S^1$ gives for elements $e_1, \ldots, e_n$ with $|e_i| = 1$

$$H^\bullet(T^n, R) = \mathbb{F}[\epsilon_1]/(e_1^2) \otimes_{\mathbb{F}} \mathbb{F}[\epsilon_2]/(e_2^2) \otimes_{\mathbb{F}} \mathbb{F}[\epsilon_n]/(e_n^2) = \bigwedge_{\mathbb{F}}[e_1, \ldots, e_n].$$

This is called an exterior algebra. A

Example 1.7 (Homology ring of projective plane)**.** From Example 1.4 we have $H^\bullet(\mathbb{P}^n_{\mathbb{C}}, R) = R \oplus R \oplus \cdots \oplus R$, with n summands in degrees $0, 2, \ldots, 2n$. Set $x = (0, 1, 0, \ldots, 0)$ to be the generator of $H^2(\mathbb{P}^n_{\mathbb{C}}, R)$. It turns out similarly to the above example that

$$H^\bullet(\mathbb{P}^n_{\mathbb{C}}, R) = R[x]/(x^{n+1}), \text{ with } |x| = 2.$$

We can apply the Künneth formula to compute

$$\begin{aligned} H^\bullet(\mathbb{P}^{d_1}_{\mathbb{C}} \times \mathbb{P}^{d_2}_{\mathbb{C}} \times \cdots \times \mathbb{P}^{d_n}_{\mathbb{C}} \times, R) &\cong R[x_1]/(x^{d_1+1}) \otimes_R R[x_2]/(x^{d_2+1}) \otimes_R \cdots \otimes_R R[x_n]/(x^{d_n+1}) \\ &\cong R[x_1, \ldots, x_n]/(x_1^{d_1+1}, \ldots, x_n^{d_n+1}), \text{ with } |x_i| = 2. \end{aligned}$$

We now come to the main result that we have been building up to. Let X be an algebraic subvariaty of $P^n_{\mathbb{C}}$ and let H denote a, (general) hyperplane in $P^n_{\mathbb{C}}$. Then $X \cap H$ is a subvariety of X of real codimension two, and thus by a, standard construction in algebraic geometry represents a cohomology class $L \in H^2(X, \mathbb{C})$ called the class of a a hyperplane section.

Theorem 1.8 (Hard Lefschetz Theorem)**.** *Let X be a smooth irreducible complex projective variety cf complex dimension n (real dimension $2n$), $H^\bullet(X) = H^\bullet(X, \mathbb{C})$, and let $L \in H^2(X, \mathbb{C})$ be the class of a a hyperplane section. Then for $0 \le i \le n$ the following maps are isomorphisms*

$$L^i : H^{n-i}(X) \to H^{n+i}(X), \text{ where } L^i(x) = \underbrace{L \cup \cdots \cup L}_{L^i} \cup x.$$

Remark 1.9. The Hard Lefschetz theorem works for $H^\bullet(X, \mathbb{F})$ where $\mathbb{F}$ is any field of characteristic zero, but the conclusion of the theorem is false in positive characteristic.

The theorem above was first stated by Lefschetz, but his proof was not entirely rigorous. The first complete proof was given by Hodge. The "standard" proof today uses the representation theory of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and is due to Chern. We will see this proof later in the course. Lefschetz's original proof was only recently made rigorous by Deligne, who extended it to positive characteristic.

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Example 1.10 (The Hard Lefschetz theorem in action)**.** For $H^\bullet(P^n_{\mathbb{C}}) = \mathbb{F}[x]/(x^{n+1})$ the class of a hyperplane is $L = x^2$ and it gives the isomorphisms

$$L^i = x^{2i} : H^{n-i}(P^n_{\mathbb{C}}) = x^{n-i}\mathbb{F} \to H^{n+i}(P^n_{\mathbb{C}}) = x^{n+i}\mathbb{F}$$
$$x^{n-i}y \mapsto (x^2)^i(x^{n-i}y) = x^{n+i}y$$

Remark 1.11.

- A consequence of the Hard Lefschetz theorem is there is an isomorphism $H^{n-i}(X) \cong H^{n+i}(X)$ and so in particular $\dim_{\mathbb{C}} H^{n-i}(X) = \dim_{\mathbb{C}} H^{n+i}(X)$. This is true in general for oriented closed manifolds (not just for smooth complex algebraic varieties) and is known as Poincaré duality.
- Another consequence of the Hard Lefschetz theorem is that if a toplogical space X can be embedded in some $\mathbb{P}^n_{\mathbb{C}}$ as a complex projective variety then $H^2(X, \mathbb{C}) \neq 0$.

Example 1.12. The sphere $S^2 = \mathbb{P}^1_{\mathbb{C}}$ is a complex projective variety known as the Riemann sphere. However whenever $n \neq 2$ in $H^\bullet(S^n)$ the class of a hyperplane section is zero since $H^2(S^n) = 0$. This shows that spheres other than the Riemann sphere are not complex projective varieties. However they still satisfies Poincaré duality since for $H^\bullet(S^n) = R[\epsilon]/(e^2)$ there are isomorphisms

$$H^0(S^n) = \mathbb{C} \cong \mathbb{C} = H^n(S^n)$$
$$H^i(S^n) = 0 \cong 0 = H^{n-i}(S^n) = 0, \quad 1 \leq i \leq \lceil n/2 \rceil.$$

Cohomology rings of n-dimesional complex projective varieties X with coefficients in a field $\mathbb{F}$ satisfy the following properties:

(1) $H^\bullet(X, \mathbb{F})$ is a graded commutative ring in the sense of (1.0.1); its even part $A := H^{2\bullet(}(X, \mathbb{F}) = \bigoplus_{i \geq 0} H^{2i}(X, \mathbb{F})$ is a commutative graded ring as defined in the next chapter. We can re-grade this ring by setting $|x| = i$ if $x \in H^{2i}(X, \mathbb{F})$. With this convention $|L| = 1$.

(2) $H^\bullet(X, \mathbb{F})$ and A are finite dimensional $\mathbb{F}$-vecor spaces (so A is an artinian ring)

(3) $H^\bullet(X, \mathbb{F})$ and A satisfiy Poincaré duality (hence A is a Gorenstein ring).

The main objective of this course is to extend the Hard Lefschetz theorem (and some weaker versions) to arbitrary rings which may not necessarily be cohomology rings but still satisfy at least some of the properties above. Thus we are motivated by the following

Question 1.13. *Which commutative graded rings A satisfying properties (1) and (2) or properties (1), (2) and (3) above also satisfy the conclusion of the Hard Lefschetz theorem?*

Chapter 2

The algebraic Lefschetz properties

From now on all rings will be commutative unless specified otherwise.

2.1 Graded artinian algebras

Definition 2.1. A commutative ring A is an ($\mathbb{N}$–)*graded ring* provided it decomposes as

$$A = \bigoplus_{i \geq 0} A_i$$

with A_i abelian groups such that $\forall i, j \in \mathbb{N}$ $A_i A_j \subseteq A_{i+j}$ ($a \in A_i, b \in A_j \Rightarrow ab \in A_{i+j}$).

Example 2.2. $A = \mathbb{F}[x_1, \ldots, x_n]$ is the fundamental example of a graded ring with $A_i =$ the set of homogeneous polynomials of degree i. Note that the degree of x_i is allowed to be an arbitrary natural number.

Remark 2.3.

- Elements of A_i are called *homogeneous elements of degree* i.
- The definition of a graded ring implies that A_0 is a subring of A and A_i are modules over A_0, so A is a module over its subring A_0. We summarize this by saying that A is an A_0-*algebra*. Quite often for us A_0 will be a field.
- A graded algebra A is called *standard graded* if A is generated by its homogeneous elements of degree one as an A_0 algebra. We write this as $A = A_0[A_1]$.

Definition 2.4. An ideal I of a graded ring is *homogeneous* if and only if I can be generated by homogeneous elements.

Example 2.5. If A is a graded ring with $A_0 = \mathbb{F}$ a field then the set of elements of positive degree

$$\mathfrak{m} = \bigoplus_{i > 0} A_i$$

is a homogeneous ideal. This is the (unique) *homogeneous maximal ideal* of A. If $A = \mathbb{F}[x_1, \ldots, x_n]$ then $\mathfrak{m} = (x_1, \ldots, x_n)$.

Exercise 2.6. Any graded Noetherian ring A is a quotient of a polynomial ring with coefficients in A_0 by a homogeneous ideal I of this polynomial ring, i.e.

$$A = \frac{A_0[x_1, \dots, x_n]}{I}.$$

Definition 2.7. The *embedding dimension* of a Noetherian graded $\mathbb{F}$-algebra A is the least integer n such that A can be written as a quotient of a polynomial ring in n variables as above.

From now on we restrict to graded rings A with $A_0 = \mathbb{F}$ a field. I will refer to these as $\mathbb{F}$-*algebras*. Note that in particular such A and each of its graded components A_i is an $\mathbb{F}$ vector space. Furthermore, if A is Noetherian then the exercise above implies that $\dim_{\mathbb{F}} A_i$ is finite for each i.

Definition 2.8. The *Hilbert function* of a graded $\mathbb{F}$-algebra A is the function

$$h_A : \mathbb{N} \to \mathbb{N}, h_A(i) = \dim_{\mathbb{F}} A_i.$$

The *Hilbert series* of A is $H_A(t) = \sum_{i\geq 0} h_A(i)t^i$.

Example 2.9. Consider $\mathbb{F}$ a field and let $A = \mathbb{F}[x, y, z]/(x^2, y^2, z^2)$. Clearly, A is a finite dimensional $\mathbb{F}$-vector space with basis given by the monomials $\{1, x, y, z, xy, y, z, xz, xyz\}$. We see that the elements of A have only four possible degrees 0,1,2,3 and moreover

$$\begin{aligned}
A_0 &= \operatorname{Span}_{\mathbb{F}}\{1\} \cong \mathbb{F} \Rightarrow h_A(0) = 1 \\
A_1 &= \operatorname{Span}_{\mathbb{F}}\{x, y, z\} \cong \mathbb{F}^3 \Rightarrow h_A(1) = 3 \\
A_2 &= \operatorname{Span}_{\mathbb{F}}\{xy, yz, xz\} \cong \mathbb{F}^3 \Rightarrow h_A(2) = 3 \\
A_3 &= \operatorname{Span}_{\mathbb{F}}\{xyz\} \cong \mathbb{F} \Rightarrow h_A(3) = 1 \\
A_i &= 0, \forall i \geq 4 \Rightarrow h_A(i) = 0, \forall i \geq 4
\end{aligned}$$

Thus $H_A(t) = 1 + 3t + 3t^2 + t^3$.

Definition 2.10. A (local or) graded $\mathbb{F}$-algebra $(A, \mathfrak{m}, \mathbb{F} = A/\mathfrak{m})$ is *artinian* if there exists a descending sequence of ideals

$$A = \mathfrak{a}_0 \supseteq \mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \cdots \supseteq \mathfrak{a}_\ell = 0 \text{ such that } \mathfrak{a}_{i-1}/\mathfrak{a}_i \cong \mathbb{F}.$$

Such a sequence of ideals is called a *composition series*. It turns out that ℓ does not depend on the choice of the composition series. The integer ℓ is called the *length* of A denoted by $\lambda(A)$.

Exercise 2.11. If A is an artinian (local or) graded $\mathbb{F}$-algebra then $\lambda(A) = \dim_{\mathbb{F}}(A)$.

Example 2.12 (Truncated polynomial ring)**.** Take $R = \mathbb{F}[x_1, \ldots, x_n]$, $\mathfrak{m} = (x_1, \ldots, x_n)$ and $A = R/\mathfrak{m}^k$ for some $k \geq 0$. We can build a composition series for A as follows: start first with the sequence of ideals

$$A \supseteq \mathfrak{m}/\mathfrak{m}^k \supseteq \mathfrak{m}^2/\mathfrak{m}^k \supseteq \cdots \supseteq \mathfrak{m}^{k-1}/\mathfrak{m}^k \supseteq \mathfrak{m}^k/\mathfrak{m}^k = 0.$$

By the third isomorphism theorem, $\left(\mathfrak{m}^{i-1}/\mathfrak{m}^k\right)/(\mathfrak{m}^i/\mathfrak{m}^k) \cong \mathfrak{m}^{i-1}/\mathfrak{m}^i \cong \mathbb{F}^{\binom{n+i-2}{i-1}}$ as R-modules. Picking an $\mathbb{F}$-basis $\{b_1, \ldots, b_u\}$ for $\left(\mathfrak{m}^{i-1}/\mathfrak{m}^k\right)/(\mathfrak{m}^i/\mathfrak{m}^k)$ allows one to refine the above sequence by inserting

$$\mathfrak{m}^{i-1}/\mathfrak{m}^k = (b_1, \ldots, b_u) \supseteq (b_1, \ldots, b_{u-1}) + (\mathfrak{m}^i/\mathfrak{m}^k) \supseteq \cdots \supseteq (b_1) + (\mathfrak{m}^i/\mathfrak{m}^k) \supseteq \mathfrak{m}^{i-1}/\mathfrak{m}^k.$$

This shows that $A = R/\mathfrak{m}^k$ is artinian.

Here are some basic properties of graded artinian rings.

Fact 2.13.
- *An $\mathbb{F}$ algebra A is artinian if and only if $\dim_{\mathbb{F}}(A) < \infty$ so artinian $\mathbb{F}$-algebras = finite dimensional $\mathbb{F}$-algebras.*
- *Let $R = \mathbb{F}[x_1, \ldots, x_n]$ and I a homogeneous ideal of R. Then $A = R/I$ is artinian if and only if $\mathfrak{m}^k \subseteq I$ for some k.*
- *A graded $\mathbb{F}$-algebra A is artinian if and only if $A = \bigoplus_{i=0}^{c} A_i$ for some integer $c \in \mathbb{N}$ and $\dim_{\mathbb{F}} A_i < \infty, \forall i \geq 0$.*

Lecture 3 – September 3, 2019

Definition 2.14. For a graded artinian $\mathbb{F}$ algebra the maximal integer c such that $A_c \neq 0$ is called the *maximal socle degree* of A. The *socle* of A is the ideal

$$(0 :_A \mathfrak{m}) = \{x \in A \mid xy = 0, \forall y \in \mathfrak{m}\}$$

and one can see that for degree reasons $A_c \subseteq (0 :_A \mathfrak{m})$.

Example 2.15. Continuing with example 2.9, thee socle degree of A is 3 and the socle is $(0 :_A \mathfrak{m}) = \mathrm{Span}\{xyz\}$, a 1-dimensional $\mathbb{F}$-vector space. This shows that A is Gorenstein. Notice that the sequence of coefficients of $H_A(t)$ is symmetric, which is a form of Poincaré duality implied by the Gorenstein property.

2.2 Lefschetz properties

2.2.1 Weak Lefschetz property and consequences

Definition 2.16. . Let $A = \bigoplus_{i=1}^{c} A_i$ be a graded artinian $\mathbb{F}$-algebra. We say that A has the ***weak Lefschetz property (WLP)*** if there exists an element $L \in A_1$ such that the multiplication map $\times L : A_i \to A_{i+1}, x \mapsto Lx$ has rank equal to $\min\{h_A(i), h_A(i+1)\}$ for all $0 \leq i \leq c-1$. We call L with this property a ***weak Lefschetz element***.

Remark 2.17 (Alternate ways to state WLP).

- Since $h_A(i)$ and $h_A(i+1)$ are the dimensions of the domain and the target of the linear transformation $\times L$ respectively, the WLP says that $\times L$ has the maximum possible rank, which is referred to as *full rank*.
- Having the WLP can be also phrased as saying that the map $\times L : A_i \to A_{i+1}$ is either injective or surjective for all $0 \le i \le c-1$.
- Finally, A has the WLP if and only if $\dim_{\mathbb{F}}(LA_i) = \min\{h_A(i), h_A(i+1)\}$ for all $0 \le i \le c-1$.

Example 2.18. Take $A = \mathbb{C}[x,y]/(x^2,y^2)$ with the standard grading $|x| = |y| = 1$ and $L = x+y$. Then the multiplication map $\times L$ gives the following matrices with respect to the monomial bases $\{1\}, \{x,y\}$ and $\{xy\}$:

map	matrix	rank	inj/ surj
$A_0 \to A_1$	$\begin{bmatrix}1\\1\end{bmatrix}$	1	inj
$A_1 \to A_2$	$\begin{bmatrix}1 & 1\end{bmatrix}$	1	surj
$A_i \to A_{i+1}, i \ge 2$	$\begin{bmatrix}0\end{bmatrix}$	0	surj

We conclude that A has the WLP and $x+y$ is a Lefschetz element on A.

Example 2.19 (Dependence on characteristic)**.** Take $A = \mathbb{F}[x,y,z]/(x^2,y^2,z^2)$ with the standard grading $|x| = |y| = 1$ and $L = ax+by+cz$. Then the multiplication map $\times L$ is represented by the following matrix with respect to the monomial bases $\{x,y,z\}$ for A_1 and $\{xy,xz,yz\}$ for A_2:

$$\times L : A_1 \to A_2 \qquad M = \begin{bmatrix} b & a & 0 \\ c & 0 & a \\ 0 & c & b \end{bmatrix}, \qquad \det(M) = -2abc.$$

This map has full rank for iff $\operatorname{char}(\mathbb{F}) \neq 2$ and $a \neq 0, b \neq 0, c \neq 0$. We conclude that A has the WLP iff $\operatorname{char}(\mathbb{F}) \neq 2$ because in that case e.g. $L = x+y+z$ is a weak Lefschetz element.

The *non-(weak) Lefschetz locus* of A in this example is

$$\begin{aligned} \mathrm{NLL}_w(A) &= \{(a,b,c) \in \mathbb{F}^3 \mid L = ax+by+cz \text{ is not a weak Lefschetz element on } A\} \\ &= V(abc) = \{(a,b,c) \in \mathbb{F}^3 \mid a = 0 \text{ or } b = 0 \text{ or } c = 0\} \\ &= \text{ the union of the three coordinate planes in } \mathbb{F}^3. \end{aligned}$$

Definition 2.20. A sequence of numbers $h_1, \ldots, h_c$ is called *unimodal* if there is an integer j such that

$$h_1 \le h_2 \le \cdots \le h_j \ge h_{j+1} \ge \cdots \ge h_c.$$

Lemma 2.21. *If B is a standard graded $\mathbb{F}$-algebra and $B_j = 0$ for some $j \in \mathbb{N}$ then $B_i = 0$ for all $i \geq j$.*

Proof. B standard graded means that $B = \mathbb{F}[B_1] = \mathbb{F}[x_1, \dots, x_n]/I$ where $x_1, \dots, x_n$ are an $\mathbb{F}$-basis for B_1 so $|x_1| = \cdots = |x_n| = 1$ and I is a homogeneous ideal.

Then we see that $B_i = \mathrm{Span}_{\mathbb{F}}\{B_{i-j}B_j\} = \mathrm{Span}_{\mathbb{F}}\{0\} = 0$ for any $i \geq j$. □

Proposition 2.22 (Prop 3.2 in book)**.** *Suppose that A is a standard graded artinian algebra over a field $\mathbb{F}$. If A has the weak Lefschetz property then A has a unimodal Hilbert function.*

Proof. Let j be the smallest integer such that $\dim_{\mathbb{F}} A_j > \dim_{\mathbb{F}} A_{j+1}$ and let L be a Lefschetz element on A. Then $\times L : A_j \to A_{j+1}$ is surjective i.e. $LA_j = A_{j+1}$. Now consider the cokernel $A/(L)$ of the map

$$A \xrightarrow{\times L} A.$$

We have that $(A/(L))_{j+1} = A_{j+1}/LA_j = 0$, so by the previous Lemma $(A/(L))_{i+i} = A_{i-j}\,(A/(L))_{j+1} = 0$ for $i \geq j$. This means that $\times L : A_i \to A_{i+1}$ is surjective for $i \geq j$ and so we have

$$h_0(A) \leq h_1(A) \leq \cdots \leq h_j(A) > h_{j+1}(A) \geq h_{j+2}(A) \geq \cdots \geq h_c(A).$$

□

Remark 2.23. The proof says that for a standard graded artinian algebra A there exists $j \in \mathbb{N}$ such that the multiplications maps by a weak Lefschetz element $\times L : A_i \to A_{i+1}$ are injective for $i < j$ after which they become surjective for $i \geq j$.

Example 2.24 (Dependence on grading)**.** Recall from Example 2.18 that the algebra $A = \mathbb{F}[x,y]/(x^2,y^2)$ with $|x| = |y| = 1$ is standard graded and has WLP and notice that the Hilbert function of A, $1,2,1$ is unimodal.

Take $B = \mathbb{C}[x,y]/(x^2,y^2)$ with $|x| = 1, |y| = 3$. Then B is a graded algebra with nonunimodal Hilbert function $1,1,0,1,1$, but x is a weak Lefschetz element on B.

Take $C = \mathbb{C}[x,y]/(x^2,y^2)$ with $|x| = 1, |y| = 2$. Then C has a unimodal Hilbert function $1,1,1,1$ but does not have the WLP.

Lecture 4 –September 5, 2019

Proposition 2.25 (Prop 3.5 in book)**.** *Let $A = \bigoplus_{i=0}^{c} A_i$ be a graded artinian $\mathbb{F}$-algebra with unimodal Hilbert function, and let $L \in A_1$. The following are equivalent:*

1. *L is a weak Lefschetz element for A.*
2. $\dim_{\mathbb{F}}(A/LA) = \max\limits_{i\geq 0}\{\dim_{\mathbb{F}} A_i\}$.
3. $\dim_{\mathbb{F}} LA = \sum_{i=0}^{c-1} \min\{\dim_{\mathbb{F}} A_i, \dim_{\mathbb{F}} A_{i+1}\}$

Definition 2.26. The numbers which appear in the proposition are called the Sperner and co-Sperner numbers of A:

$$\mathrm{Sperner}(A) = \max_i\{\dim_{\mathbb{F}} A_i\} \tag{2.2.1}$$

$$\mathrm{CoSperner}(A) = \sum_{i=0}^{c-1} \min\{\dim_{\mathbb{F}} A_i, \dim_{\mathbb{F}} A_{i+1}\}. \tag{2.2.2}$$

Definition 2.27. In a (local or) graded Noetherian ring (which includes artinian local or graded rings) a set $\{f_1, \dots, f_n\}$ is a minimal generating set for an ideal I if $I = (f_1, \dots, f_n)$ and no proper subset of $\{f_1, \dots, f_n\}$ generates I.

Fact 2.28 ((Graded) Nakayama Lemma)**.** *In a (local or) graded Noetherian ring* $(R, \mathfrak{m}, \mathbb{F})$ *all minimal generating sets of a given ideal I have the same cardinality, called the* minimal number of generators *of I and denoted $\mu(I)$. Furthermore $\mu(I) = \dim_{\mathbb{F}} I/\mathfrak{m}I$.*

Definition 2.29. The *Dilworth number* of an artinian algebra A, denoted $d(A)$, is defined to be the supremum of the cardinalities of minimal generating sets of all ideals in A, i.e.

$$d(A) = \sup\{\mu(I) \mid I \text{ ideal of } A\}.$$

Definition 2.30. An artinian Algebra is said to satisfy the ***Sperner property*** if the Dilworth number of A is equal to the Sperner number of A.

Remark 2.31.
- If $A = \mathbb{F}[x_1, \dots, x_n]/I$ is the quotient of a polynomial ring by an ideal I generated by monomials, then a standard basis for A as an $\mathbb{F}$ vector space consists of the monomials not belonging to I. These form a poset $\mathcal{B}$ with respect to divisibility. In this context, $d(A)$ is equal to the size of the largest antichain (set of pairwise incomparable monomials) in $\mathcal{B}$, which is shows that the definition of the Dilworth number above recovers the notion by the same name from poset combinatorics. (In combinatorics, the Dilworth number of a poset is the size of the largest antichain.)
- Sperner's Thorem from combinatorics can be restated to say that the artinian algebra $\mathbb{F}[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)$ or equivalently the boolean lattice $2^{[n]}$ satisfies the Sperner property.

Proposition 2.32 (Prop 3.6 in book)**.** *If A is a standard graded artinian algebra which has the WLP then A satisfies the Sperner property.*

While I won't present the proof notice that one inequality is always true for standard graded A. Since

$$\mu(\mathfrak{m}^j) = \dim_{\mathbb{F}} \mathfrak{m}^j/\mathfrak{m}^{j+1} = \dim_{\mathbb{F}} \bigoplus_{i=j}^{c} A_i / \bigoplus_{i=j+1}^{c} A_i = \dim_{\mathbb{F}} A_j$$

we have $d(A) \geq \sup_{j\geq 0}\{\mu(\mathfrak{m}^j)\} = \sup_{i\geq 0}\{\dim_{\mathbb{F}}(A_j)\} = \mathrm{Sperner}(A)$. So the WLP is really useful for establishing the opposite inequality.

2.2.2 Strong Lefschetz property and consequences

Definition 2.33. . Let $A = \bigoplus_{i=1}^{c} A_i$ be a graded artinian $\mathbb{F}$-algebra. We say that A has the ***strong Lefschetz property (SLP)*** if there exists an element $L \in A_1$ such that the multiplication maps $\times L^d : A_i \to A_{i+d}, x \mapsto L^d x$ has rank equal to $\min\{h_A(i), h_A(d+i)\}$ for all $1 \le d \le c$ and $0 \le i \le c-d$. We call L with this property a ***strong Lefschetz element***.

Remark 2.34 (SLP $\Rightarrow$ WLP). If A satisfies the SLP then A satisfies the WLP, which is the $d = 1$ case.

Example 2.35 (Dependence on characteristic)**.** Take $A = \mathbb{F}[x,y]/(x^2,y^2)$ with the standard grading $|x| = |y| = 1$ and $L = ax + by$. Then the multiplication map $\times L^2$ gives the following matrices with respect to the monomial bases $\{1\}, \{x,y\}$ and $\{xy\}$:

map	matrix	rank	inj/ surj
$A_0 \to A_2$	$\begin{bmatrix}2ab\end{bmatrix}$	$\begin{cases}1 & \mathrm{char}(\mathbb{F}) \neq 2\\ 0 & \mathrm{char}(\mathbb{F}) \neq 2\end{cases}$	$\begin{cases}\text{bij} & \mathrm{char}(\mathbb{F}) \neq 2\\ \text{none} & \mathrm{char}(\mathbb{F}) \neq 2\end{cases}$
$A_i \to A_{i+2}, i \ge 1$	$\begin{bmatrix}0\end{bmatrix}$	0	surj

If $\mathrm{char}(\mathbb{F}) \neq 2$ we conclude that A has the SLP and $ax + by$ where $a \neq 0, b \neq 0$ is a Lefschetz element on A. The *non-(strong) Lefschetz locus* is the union of the coordinate axes in $\mathbb{F}^2$

$$NLL_s(A) = V(ab) = \{(a,b) \in \mathbb{F}^2 \mid a = 0 \text{ or } b = 0\}.$$

However A does not have the SLP if $\mathrm{char}(\mathbb{F}) = 2$ so in that case $\mathrm{NLL}_s(A) = \mathbb{F}^2$.

Proposition 2.36 (Prop 3.9 in book)**.** *Let A be a (not necessarily standard) graded artinian $\mathbb{F}$-algebra which satisfies the SLP. Then A has unimodal Hilbert function.*

Proof. Suppose that the Hilbert function of A is not unimodal. Then there are integers $k < l < m$ such that $\dim_{\mathbb{F}} A_k > \dim_F A_l < \dim_{\mathbb{F}} A_m$. Hence the multiplication map $\times L^{m-k} : A_k \to A_m$ cannot have full rank for any linear element $L \in A$ because it is the composition of $\times L^{m-l} : A_l \to A_m$ and $\times L^{l-k} : A_l \to A_k$, each of which have rank strictly less than $\min\{\dim_{\mathbb{F}} A_k, \dim_{\mathbb{F}} A_m\}$. Thus A cannot have the SLP. □

Definition 2.37. . Let $A = \bigoplus_{i=1}^{c} A_i$ be a graded artinian $\mathbb{F}$-algebra. We say that A has the ***strong Lefschetz property in the narrow sense (SLPn)*** if there exists an element $L \in A_1$ such that the multiplication maps $\times L^{c-2i} : A_i \to A_{c-i}, x \mapsto L^{c-2i}x$ are bijections for all $0 \le i \le \lceil c/2 \rceil$.

Remark 2.38. SLP in the narrow sense is the closest property to the conclusion of the Hard Lefschez Theorem.

Definition 2.39. We say that a graded artinian algebra $A = \bigoplus_{i=1}^{c} A_i$ of maximum socle degree c has a *symmetric Hilbert function* if $h_A(i) = h_A(c-i)$ for $0 \le i \le \lceil c/2 \rceil$.

Proposition 2.40. *If a graded artinian $\mathbb{F}$-algebra A has the strong Lefschetz property in the narrow sense, then the Hilbert function of A is unimodal and symmetric. Moreover we have the equivalence:*

$$A \text{ has SLP + symmetric Hilbert function} \Leftrightarrow A \text{ has SLP in the narrow sense.}$$

Proof. ($\Leftarrow$) The fact that SLP in the narrow sense implies symmetric Hilbert function follows from the definition because the bijections give $\dim_F A_i = \dim_F A_{c-i}$.

The fact that SLP in the narrow sense implies SPL can be noticed by considering $\times L^d : A_i \to A_{i+d}$. For each such d, i there exists $j = c - i - d$ such that:

- if $i \le (c-d)/2$ then $j = c-i-d \le i$ and $(\times L^d)\circ(\times L^{j-i}) = \times L^{c-2i}$ is a bijection implies that $\times L^d$ is surjective, hence has full rank;
- if $i > (c-d)/2$ then $c-i > d+i$ and $(\times L^{j-i})\circ(\times L^{c-d-2i}) = \times L^{c-2i}$ is a bijection implies that $\times L^d$ is injective, hence full rank;

($\Rightarrow$) The fact that SLP + symmetric Hilbert function implies SLPn is clear from the definitions. □

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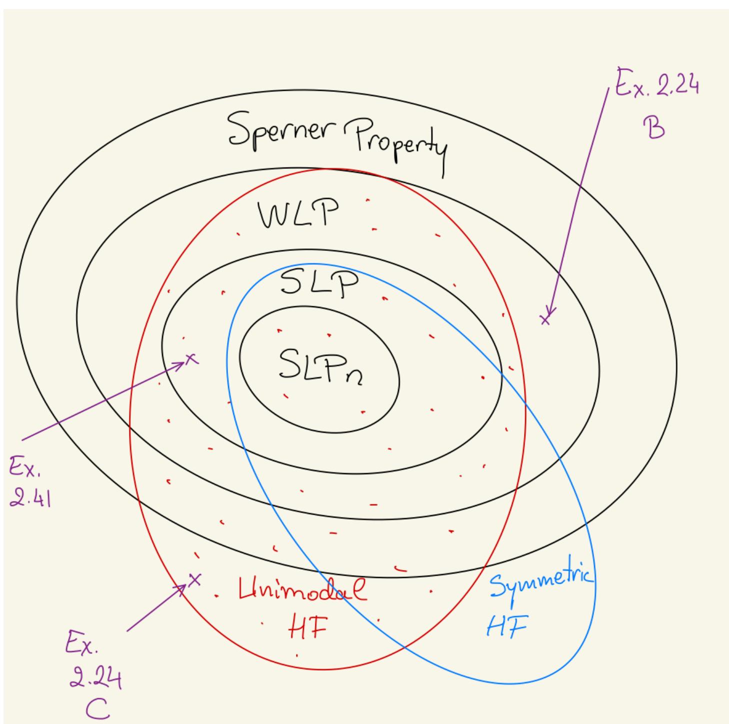
Example 2.41. The algebra $\mathbb{F}[x,y]/(x^2, xy, y^a)$ with $a > 3$ has non-symmetric Hilbert function $1, 2, \underbrace{1, \ldots, 1}_{a-2}$. Notice that A has the SLP because $L = x+y$ is a strong Lefschetz element, A does not satisfy SLPn because its Hilbert function is not symmetric.

Here is a summary of what we know so far. Note that the (SLP$\cap$ symmetric HF) $\setminus$ SLPn is region is empty but all the other regions are probably not.



2.2.3 Stanley's Theorem

The most famous theorem in the area of investigation of the algebraic Lefschetz properties, and also the theorem which started this, is the following:

Theorem 2.42 (Stanley's theorem)**.** *If* $\mathrm{char}(\mathbb{F}) = 0$*, then all monomial complete intersections, i.e.* $\mathbb{F}$*-algebras of the form*

$$A = \frac{\mathbb{F}[x_1, \ldots, x_n]}{(x_1^{d_1}, \ldots, x_n^{d_n})}$$

with $d_1, \ldots, d_n \in \mathbb{N}$ *have the SLP.*

Proof. Recall that $H^\bullet(\mathbb{P}^{d-1}_{\mathbb{C}}, \mathbb{F}) = \mathbb{F}[x]/(x^d)$, so by Künneth we have

$$H^\bullet(\mathbb{P}^{d_1-1}_{\mathbb{C}} \times \mathbb{P}^{d_2-1}_{\mathbb{C}} \times \cdots \times \mathbb{P}^{d_n-1}_{\mathbb{C}}, \mathbb{F}) = \mathbb{F}[x]_1/(x_1^{d_1}) \otimes_{\mathbb{F}} \mathbb{F}[x_2]/(x_2^{d_2}) \otimes_k \cdots \otimes_k \mathbb{F}[x_n]/(x_n^{d_n}) = A.$$

Since $X = \mathbb{P}^{d_1-1}_{\mathbb{C}} \times \mathbb{P}^{d_2-1}_{\mathbb{C}} \times \cdots \times \mathbb{P}^{d_n-1}_{\mathbb{C}}$ is an irreducible complex projective variety, the Hard Lefschetz theorem says that A has SLP in the narrow sense which implies that A has SLP. □

We will give several more proofs of Stanley's theorem in this class.

2.3 Jordan type

Fact 2.43. *If* $\mathbb{F}$ *is algebraically closed,* V *is a finite dimensional* $\mathbb{F}$ *vector space and* $T : V \to V$ *is a linear transformation, then there is a basis* $\mathcal{B}$ *for* V *with respect to which* T *is represented by a matrix in* Jordan canonical form *(JCF)*

$$[T]_{\mathcal{B}} = \begin{bmatrix} J_{p_1}(\lambda_1) & 0 & \cdots & 0 \\ 0 & J_{p_2}(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{p_t}(\lambda_t) \end{bmatrix},$$

where $\lambda_1, \ldots, \lambda_t$ *are the (not necessarily distinct) eigenvalues of* T *and the matrices*

$$J_{p_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 0 & \cdots & 0 & 0 \\ 1 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \lambda_i \end{bmatrix}$$

of size $p_i \times p_i$ *are called the Jordan blocks of* T*.*

Taking V to be an artinian algebra and $L \in A_1$ we will seek the JCF of the map $\times L : A \to A, a \mapsto La$.

Lemma 2.44. *If A is an artinian algebra and $L \in A_1$ the only eigenvalue for the linear transformation $T = \times L : A \to A, a \mapsto La$ is zero.*

Proof. Note that if c is the socle degree of A then $T^{c+1} = \times L^{c+1} = 0$. Thus the minimal polynomial of T divides X^{c+1}, so it is a power of X. Then since the eigenvalues are roots of the minimal polynomial and 0 is the only root of the minimal polynomial of T the conclusion follows. □

Definition 2.45. The *Jordan type* of the multiplication map $T = \times L : A \to A$, denoted $P_A(\times L)$ is the sequence of sizes of the Jordan blocks of T, written in weakly decreasing order. This is a partition for $\lambda(A)$ since the sizes of the Jordan blocks sum to $\lambda(A)$.

Definition 2.46. The *Hilbert function partition* H_A of an artinian $\mathbb{F}$-algebra A is the sequence of positive values of the Hilbert function of A written in weakly decreasing order. This is a partition for $\lambda(A)$ since the sizes of the Jordan blocks sum to $\lambda(A)$.

One often represents partitions graphically as Young diagrams.

Example 2.47. Say $A = \mathbb{F}[x, y]/(x^2, y^2), L = x + y$ and consider the basis $\mathcal{B} = \{1, x + y, xy, x - y\}$ for A. Then

$$[\times L]_{\mathcal{B}} = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

and so $P_A(\times L) = (3, 1)$, while $H_A = (2, 1, 1)$.

Definition 2.48. Given a partition P written down as a Young diagram, the *dual partition* $P^\vee$ is the partition obtained by transposing the Young diagram of P.

Proposition 2.49 (Dual partition to the Jordan type)**.** *Given a linear transformation $T : V \to V$ with $\dim_{\mathbb{F}} V < \infty$ the dual partition for the partition of $\dim V$ by the sizes of the Jordan blocks of T is given by the integers*

$$e_i = \dim_{\mathbb{F}} \operatorname{Ker}(T^i) - \dim_F \operatorname{Ker}(T^{i-1}), i \geq 1.$$

In particular if $V = A$ an artinian algebra and $T = \times L$ then $e_i = \dim_F \frac{(0:_A L^i)}{(0:_A L^{i-1})}$.

Proof. Note that the kernels form an increasing sequence of subspaces of V, a finite dimensional vector space, hence this sequence will eventually stabilize.

$$0 = \operatorname{Ker}(T^0) \subseteq \operatorname{Ker}(T^1) \subseteq \cdots \subseteq \operatorname{Ker}(T^s) = \operatorname{Ker}(T^{s+1}) = \cdots$$

This shows that $e_i \geq 0$ and $e_i = 0$ for $i \gg 0$ (in fact $i > s$).

The rigorous proof then proceeds by induction on i. I will only sketch the proof of why e_1, e_2 are the first two values of the dual partition for the Jordan type $P(T)$.

Note that the first part of $P(T)$ is the number of Jordan blocks of T. These blocks are in bijection with a basis for $\operatorname{Ker}(T)$ since each Jordan block contains a unique zero column. This shows that $P(T)_1^\vee = e_1$.

The second part of $P(T)$ is the number of Jordan blocks of T of size at least 2. Notice that each block of size $p \geq 2$ for T gives two blocks for T^2 one of size 1 and another of size $p-1$. The blocks of T^2 are in bijection with a basis for $\operatorname{Ker}(T^2)$. This shows that

$$\dim_{\mathbb{F}} \operatorname{Ker}(T^2) = \#\text{ blocks of } T^2 = \#\text{ blocks of } T + \#\text{ blocks of } T \text{ of size } \geq 2.$$

Hence

$$e_2 = \dim_{\mathbb{F}} \operatorname{Ker}(T^2) - \dim \operatorname{Ker}_{\mathbb{F}} T = \#\text{ blocks of } T \text{ of size } \geq 2 = P(T)_2^\vee.$$

□

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Definition 2.50. There is a partial order on partitions called *dominance order* where if $P = (p_1 \geq p_2 \geq \cdots \geq p_r), Q = (q_1 \geq q_2 \geq \cdots \geq q_s)$ are partitions of a positive integer n then $P \trianglelefteq Q \iff \sum_{i=1}^k p_i \leq \sum_{i=1}^k q_i \leq$ for $k \geq 1$.

Definition 2.51. Consider a graded artinian $\mathbb{F}$-algebra A and $L \in \mathfrak{m}$. Recall that there is an ordered basis $\mathcal{B}$ such that

$$[\times L]_{\mathcal{B}} = \begin{bmatrix} J_{p_1}(0) & 0 & \cdots & 0 \\ 0 & J_{p_2}(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{p_t}(0) \end{bmatrix}.$$

Denote the 1-st, p_1+1-st, ..., p_t+1-st elements of $\mathcal{B}$ by $b_1, \ldots, b_t$. The elements of $\mathcal{B}$ can be enumerated as

$$\begin{aligned} S_1 &= \{b_1, Lb_1, L^2b_1, \ldots, L^{p_1-1}b_1\} \\ S_2 &= \{b_2, Lb_2, L^2b_2, \ldots, L^{p_2-1}b_2\} \\ &\cdots \\ S_t &= \{b_t, Lb_t, L^2b_2, \ldots, L^{p_t-1}b_t\} \end{aligned}$$

and we call these lists the t *strands* of the map $\times L : A \to A$ and the number $|S_i|$ are the lengths of the strands.

Remark 2.52.
- We can choose the b_i to be homogeneous. In that case, every element of $\mathcal{B}$ will be homogeneous too and the homogeneous elements of $\mathcal{B}$ of degree i will be a basis for A_i for each $i \geq 0$.

- The elements of each strand have distinct degrees.
- With the notation above $P_A(L) = (p_1, p_2, \dots, p_t)$, where p_i is the length of the i-th longest strand of the multiplication map $\times L : A \to A$.

We can now give criteria for L to be a strong/weak Lefschetz element on A in terms of $P_A(L)$. To do this we need a lemma first.

Lemma 2.53 (Lemma 3.65 in book)**.** *Let A be an artinian $\mathbb{F}$-algebra with unimodal Hilbert function and suppose L is a weak Lefschetz element on A so that the image $\bar{L}$ of L in $B = A/(0 :_A L)$ is strong Lefschetz, then y is strong Lefschetz on A.*

Theorem 2.54 (Jordan type theorem)**.** *Let A be a standard graded artinian $\mathbb{F}$ algebra with unimodal Hilbert function and let $L \in A_1$. Then*

1. *the number of Jordan blocks of $\times L$ is* $\geq \text{Sperner}(A)$
2. *L is a weak Lefschetz element of A $\iff$ the number of Jordan blocks of $\times L$ is equal to* $\text{Sperner}(A)$
3. *$P_A(\times L) \trianglelefteq H_A^\vee$ in dominance order*
4. *L is a strong Lefschetz element of A $\iff$ $P_A(\times L) = H_A^\vee$*

Proof. Suppose a basis $\mathcal{B}$ has been chosen such that $[\times L]_{\mathcal{B}}$ is in JCF.

1. Follows because the strands of $\times L$ have elements in distinct degrees, thus the $\text{Sperner}(A)$ elements of $\mathcal{B}$ which span the largest graded piece of A are divided among at least as many strands and the number of strands ins the number of Jordan blocks.

2. Set $K = \text{Ker}(\times L)$. Notice that each strand has a unique element in K, $S_i \cap K = \{b_i L^{p_i - 1}\}$. Thus the number of Jordan blocks is $\dim_{\mathbb{F}} K$. From proposition 3.1 it follows that L is a weak Lefschetz element of A $\iff$ $\dim_{\mathbb{F}} K = \dim_{\mathbb{F}}(A/LA) = \text{Sperner}(A)$.

3. Let $Q = H_A^\vee$ be the dual partition of H_A and notice that if $Q = (q_1 \geq q_2 \geq \cdots \geq q_s)$ then q_i is the number of parts of H_A of size at least i. Thus

$$\begin{aligned}
\sum_{k=1}^{i} q_k &= \sum_{k=1}^{i} \#\text{ parts of } H_A \text{ of size } \geq k \\
&= \#\text{parts of } H_A \text{ of size } 1 + 2(\#\text{parts of } H_A \text{ of size } 2) + \cdots + i(\#\text{parts of } H_A \text{ of size } \geq i) \\
&= \sum_{k s.t. h_A(k) < i} h_A(k) + \sum_{k s.t. h_A(k) \geq i} i.
\end{aligned}$$

Now let $P = P_A(\times L) = (p_1 \geq p_2 \geq \cdots \geq p_r)$ and notice that

$$\begin{aligned}
\sum_{k=1}^{i} p_k &= \sum_{k=1}^{i} i \text{ largest lengths of strands of } \times L \\
&\leq \min\{h_A(0), i\} + \min\{h_A(1), i\} + \cdots + \min\{h_A(c), i\} \\
&= \sum_{k s.t. h_A(k) < i} h_A(k) + \sum_{k s.t. h_A(k) \geq i} i.
\end{aligned}$$

where the inequality can be justified by noticing that the sum of the lengths of the longest i strands is the dimension of the vector space W they span and considering the fact that $\dim(W \cap A_k)$ is less or equal to each of $\dim_{\mathbb{F}} A_k$, and the number of strands, i (the latter because each strand has at most one element in degree k).

This proves that $P \trianglelefteq Q$ in dominance order.

4. ($\Rightarrow$) Suppose L is a strong Lefschetz element.

Claim: the first i longest strands of $\times L$ span the components A_k with $h_A(k) \leq i$.

The proof is by induction on i. For $i = 1$ since $L^c : A_0 \to A_c$ has rank 1, the strand $1, L, L^2, \ldots, L^c$ goes through all the graded pieces of A of dimension at least 1 and spans each of them. For arbitrary i, say by unimodality that $h_A(k) \geq i$ for $k_1 \leq k \leq k_2$. Notice that there are elements in $\mathcal{B}$ of degree k_1 and k_2 that don't belong to the union of the first $i-1$ longest strands. Since $\times L^{k_2-k_2} : A_{k_1} \to A_{k_2}$ has full rank there must be $\min\{h_A(k_1), h_A(k_2)\} - i + 1$ strands of A not in the first $i-1$ longest strands which connect $A_{k_1} \to A_{k_2}$. These are necessarily the next longest strands because all remaining strands have to consist of elements of degrees between k_1 and k_2 and they must span the components A_k with $h_A(k) \leq \min\{h_A(k_1), h_A(k_2)\}$, which includes the claim for i.

From the claim, it can be seen that the strands of $\times L$ form the dual partition to H_A.

($\Leftarrow$) Perform induction on $c =$ the maximum socle degree of A. Set j to be such that $h_A(j)$ is the largest value of h_A.

Since the number of parts of $H_A^\vee$ is the Sperner number of A and the number of parts of $P_A(L)$ is the number of Jordan blocks, we see that their equality implies L is a weak Lefschetz element on A by part 2.

Now consider $A' = A/(0 :_A L)$. The exact sequence

$$0 \to (0 :_A L)(-1) \to A(-1) \xrightarrow{L} A \to A/LA \to 0$$

leads to a short exact sequence

$$0 \to A'(-1) \xrightarrow{L} A \to A/LA \to 0.$$

This allows to compute

$$\begin{aligned} h_{A'}(i) &= h_A(i+1) - h_{A/LA}(i+1) = h_{LA}(i+1) = \min\{h_A(i), h_A(i+1)\} \\ &= \begin{cases} h_A(i), & i \leq j \\ h_A(i+1), & i > j \end{cases}. \end{aligned}$$

In summary the Hilbert function of A' is

$$h_{A'} = h_A(0), h_A(1), \ldots, h_A(j-1), h_A(j+1), \ldots, h_A(c)$$

which shows that the maximum socle degree of A' is $c' = c - 1$, and

$$H_{A'}^\vee = (q_1 - 1, q_2 - 1, \ldots, q_c - 1).$$

Furthermore, thinking of a basis for A' as being obtained from a basis of A by removing the last entry in each strand yields that the JCF of multiplication by $\bar{L}$ on A' has blocks of size

$$P_{A'}(\bar{L}) = (p_1 - 1, p_2 - 1, \ldots, p_r - 1).$$

Hence $H_A^\vee = P_A(L)$ implies $H_{A'}^\vee = P_{A'}(\bar{L})$ and the inductive hypothesis implies that $\bar{L}$ is a strong Lefschetz element on A'. Since L is weak Lefschetz on A, lemma 2.53 implies that L is strong Lefschetz on A.

□

Chapter 3

Proving the Lefschetz properties algebraically

Lecture 7 – September 17, 2019

3.1 Embedding dimension two via initial ideals

Any graded artinian algebra of embedding dimension one is isomorphic to $\mathbb{F}[x]/(x^d)$ for some d, which we already know has the SLPn for $\mathbb{F}$ of any characteristic.

In this section we study the Lefschetz properties of Artinain gradde algebras of embedding dimension two, which have the form $A = \mathbb{F}[x, y]/I$, where I is a homogeneous ideal.

Theorem 3.1. *If $A = \mathbb{F}[x, y]/I$ is a graded artinian ring, then*

1. *A has unimodal Hilbert function with* $\text{Sperner}(A) = \alpha(I)$
2. *A has the WLP regardless of the characteristic of* $\mathbb{F}$
3. *A has the SLP if* $\text{char}(\mathbb{F}) > \text{socledeg(A)}$.

Proof. 1. To show the WLP, we compute the Sperner number of A. Set $\alpha = \alpha(I)$ to be the smallest degree of any nonzero element of I. Then we claim that $\text{Sperner}(A) = \dim_{\mathbb{F}} A_{\alpha-1} = \alpha$. The equalities $\dim_{\mathbb{F}} A_{\alpha-1} = \dim_{\mathbb{F}} \mathbb{F}[x, y] = \alpha$ follow from the definition for $\alpha(I)$ as the initial degree of I. Moreover, letting $f \in I$ be a homogeneous element of degree α, since $f\mathbb{F}[x, y]_{i-\alpha} \subseteq I_i$ for any $i > \alpha$ we have

$$h_A(i) \leq h_{\mathbb{F}[x,y]}(i) - \dim_{\mathbb{F}} \mathbb{F}[x, y]_{i-\alpha} = i + 1 - (i - \alpha + 1) = \alpha.$$

We may assume, after a linear change of variables, that f contains the monomial x^α. Then $A/yA = k[y]/(y^\alpha)$ has $\dim A/yA = \alpha = \text{Sperner}(A)$. From proposition we see that y is a weak Lefschetz element on A. □

To prove the assertion regarding the SLP we need some more tools.

Definition 3.2. The *graded lexicographic order* is a total order on monomials in a polynomial ring defined as follows. For monomials $m = \prod x_i{}^{a_i}$ and $m' = \prod x_i{}^{b_i}$

$$m >_{\text{rlex}} m' \iff \begin{cases} \deg m > \deg m' \text{ or} \\ \deg m = \deg m' \text{ and } a_i < b_i \text{ for the last index } i \text{ with } a_i \neq b_i. \end{cases}$$

Definition 3.3. The *leading term* of a polynomial f with respect to a given monomial ordering $<$ is $\text{lt}_<(f) =$ the largest monomial appearing in f. The *initial ideal* for an ideal I with respect to a monomial order is $\text{in}_<(I) = \{\text{lt}_<(f) \mid f \in I\}$.

Example 3.4. $x_1^3x_2x_4^3 < x_1x_2x_3x_4^4$ and $\text{lt}(x_1^3x_2x_4^3 + x_1x_2x_3x_4^4 + x_1^2x_4^5 + x_4^7) = x_1^3x_2x_4^3$

Remark 3.5. The graded reverse lexicografic order has the folowing property: if f is homogeneous and $x_n \mid \text{lt}(f)$ then $x_n \mid f$.

The initial ideal has the following nice properties:

Lemma 3.6. *Let I be a homogeneous ideal in a polynomial ring $R = \mathbb{F}[x_1, \dots, x_n]$ and set*

$$A = R/I, A' = R/\operatorname{in}_{\text{revlex}}(I).$$

1. *if $<$ is any monomial order then $\dim_{\mathbb{F}} I_i = \dim_{\mathbb{F}} \text{in}_<(I)_i$ (a theorem of Macaulay).*
2. $(\text{in}_{\text{revlex}}(I) :_R x_n) = \text{in}_{\text{revlex}}(I :_R x_n)$*, so* $\dim_{\mathbb{F}}(0 :_A: x_n)_i = \dim_{\mathbb{F}}(0 :_{A'} x_n)_i, \forall i \in \mathbb{N}$ *and similarly with x_n^d instead of x_n.*

Proof. (2.) Let $m \in (\text{in}_{\text{revlex}}(I) :_R x_n)$, then $mx_n \in \text{in}_{\text{revlex}}(I)$ is the leading term of some homogeneous polynomial g implies that $x_n \mid g$, so $h = g/x_n \in (I :_R x_n)$ and $\text{lt}(h) = m$. Conversely, if $m \in \text{in}(I :_R x_n)$ then $m = \text{lt}(h)$ and $hx_n \in I$, so $mx_n \in \text{in}(I)$, i.e. $m \in (\text{in}(I) :_R x_n)$.

For the second statement:

$$\begin{aligned} \dim_{\mathbb{F}}(0 :_A: x_n)_i &= \dim_{\mathbb{F}} \left((I :_R x_n)/I\right)_i = \dim_{\mathbb{F}}(I :_R x_n)_i - \dim_{\mathbb{F}} I_i \\ &= \dim_{\mathbb{F}} \text{in}(I :_R x_n)_i - \dim_{\mathbb{F}} \text{in}(I)_i \\ &= \dim_{\mathbb{F}}(\text{in}(I) :_R x_n)_i - \dim_{\mathbb{F}} \text{in}(I)_i \\ &= \dim_{\mathbb{F}} \left((\text{in}(I) :_R x_n)/\operatorname{in}(I)\right)_i = \dim_{\mathbb{F}}(0 :_{A'}: x_n)_i. \end{aligned}$$

□

Corollary 3.7. *With the notation in the previous Lemma, if x_n is a strong/weak Lefschetz element on A' then it is also a strong/weak Lefschetz element on A.*

Proof. Being a Lefschetz element is determined by whether

$$\dim_F(0 :_A x_n^d)_i = \min\{0, h_A(d+i) - h_A(i)\}$$

which is equivalent by the previous Lemma to

$$\dim_F(0 :_{A'} x_n^d)_i = \min\{0, h_{A'}(d+i) - h_{A'}(i)\}.$$

□

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We have reduced the proof of part 2 of theorem 3.1 to

Proposition 3.8. *If I is an initial ideal w.r.t the graded revlex order, then $A' = \mathbb{F}[x,y]/I$ has the SLP provided* $\mathrm{char}(\mathbb{F}) > \mathrm{socdeg(I)}$.

Proof. If $\mathrm{char}(\mathbb{F}) > \mathrm{socdeg(I)}$ and I is an initial ideal w.r.t the graded revlex order we may assume, after a change of variables that I is *strongly stable*, that is it has the property $x^a y^b \in I \Rightarrow x^{a+1}y^{b-1} \in I$. Define

$$a_i = \min\{a \mid x^a y^{i-a} \in I\}$$

and set by convention

$$a_0 = 1 < a_1 = 2 < \cdots < a_{\alpha(I)-1} = \alpha(I).$$

Notice that $a_i \geq a_{i+1}$ for $i \geq \alpha(I)$ so we have

$$a_0 = 1 < a_1 = 2 < \cdots < a_{\alpha(I)-1} = \alpha(I) \leq a_{\alpha(I)} \geq \cdots \geq a_c \geq a_{c+1} = 0.$$

Now notice that $\dim_{\mathbb{F}} A_i = a_i$ since $A_i = \mathrm{Span}\{x^a y^{i-a} \mid 0 \leq a \leq a_i\}$. Since $\times y^d : A_i \to A_{i+d}$ takes $x^a y^{i-a} \mapsto x^a y^{i+d-a}$ it is clear that this map is injective iff $a_i \leq a_{i+d}$ and surjective iff $a_i \geq a_{i+d}$. □

3.2 Representations of $\mathfrak{sl}_2$ and the SLPn

3.2.1 The Lie algebra $\mathfrak{sl}_2$ and its representations

Definition 3.9. A *Lie algebra* is a vector space $\mathfrak{g}$ equipped with a bilinear operator $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the following two conditions :

- $[x,y] = -[y,x]$
- $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0.$

The bilinear operator $[-,-]$ is called the bracket product, or simply the *bracket*. The second identity in the definition is called the *Jacobi identity*.

Any associative algebra has a Lie algebra structure with the bracket product defined by commutator $[x,y] = xy - yx$. The associativity implies the Jacobi identity.

Let $\mathbb{F}$ be a field of characteristic zero throughout this section.The set of $n \times n$ matrices $M_n(\mathbb{F})$ forms a Lie algebra since it is associative. This Lie algebra is denoted by $\mathfrak{gl}_n(\mathbb{F})$.

Definition 3.10. Since the set of matrices of trace zero is closed under the bracket (because $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any matrices A, B), it forms a Lie subalgebra

$$\mathfrak{sl}_n(\mathbb{F}) = \{M \in \mathfrak{gl}_n(\mathbb{F}) \mid \mathrm{tr}(M) = 0\}.$$

Example 3.11. In the case where $n = 2$, $\mathfrak{sl}_2(\mathbb{F})$ is three-dimensional, with basis

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

The three elements e, f, h are called the $\mathfrak{sl}_2$-triple.

These elements satisfy the following three relations, which we call the fundamental relations:

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f. \tag{3.2.1}$$

The algebra $\mathfrak{sl}_2(\mathbb{F})$ is completely determined by these relations.

We are interested in representations of $\mathfrak{sl}_2$.

Definition 3.12. Let V be an $\mathbb{F}$-module. A *representation* of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism

$$\rho : \mathfrak{g} \to \mathrm{End}(V),$$

i.e. a vector space homomorphism which satisfies

$$\rho([x, y]) = [\rho(x), \rho(y)].$$

A representation is called *irreducible* if it contains no trivial (nonzero) subrepresentation i.e. if $W \subsetneq V$ is such that $\rho(W) \subseteq W$ then $W = 0$.

In the case of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{F})$, we call the set of elements $E = \rho(e), H = \rho(h), F = \rho(f)$ an $\mathfrak{sl}_2$-triple since they also satisfy $[E, F] = H, [H, E] = 2E, \quad [H, F] = -2F$.

Example 3.13. For each $X \in \mathfrak{sl}_2(\mathbb{F})$, consider the linear map $\mathrm{ad}(X) : \mathfrak{sl}_2(\mathbb{F}) \to \mathfrak{sl}_2(\mathbb{F}), \mathrm{ad}(X)(Y) = [X, Y]$. This gives a representation

$$\rho : \mathfrak{sl}_2(\mathbb{F}) \to \mathrm{End}(\mathfrak{sl}_2(\mathbb{F})), X \mapsto \mathrm{ad}(X).$$

Consider $H = \mathrm{ad}(h)$ and note that equation (3.2.1) says that e, h, f are an eigenvectors of H with eigenvalues $2, 0, -2$ respectively. Hence the endomorphism H is represented by following matrix:

$$[H]_{\{e,h,f\}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Two important results on Lie algebra representations are:

Fact 3.14.

1. *Any Lie algebra representation has an irreducible subrepresentation.*
2. *[Weyl's Theorem] Any Lie algebra representation decomposes uniquely up to isomorphism as a direct sum of irreducible representations.*

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Definition 3.15. Let $\rho : \mathfrak{sl}_2(\mathbb{F}) \to \mathrm{End}(V)$ be a representation with $\rho(h) = H, \rho(e) = E, \rho(f) = F$. The eigenvalues of H are called *weights* and the eigenvectors are called *weight vectors*. In particular and an eigenvector u is called a *lowest weight vector* if $Fu = 0$ and is called a *highest weight vector* if $Eu = 0$.

To justify the name of highest weight we show:

Theorem 3.16 (Classification of representations of $\mathfrak{sl}_2$ – Lemma 3.25 in the book)**.** *Let $\rho : \mathfrak{sl}_2(\mathbb{F}) \to \mathrm{End}(V)$ be an irreducible representation with $\dim(V) = n + 1$. Then there exist a basis $\mathcal{B} = \{v_0, \dots, v_n\}$ for V such that*

1. *each v_i is an eigenvector for H with eigenvalue $-n + 2i$, i.e. $Hv_i = (-n + 2i)v_i$*
2. *$Ev_i = v_{i+1}$ for $i < n$, $Ev_n = 0$*
3. *$Fv_i = i(n - i + 1)v_{i-1}$ for $i > 0$, $Fv_0 = 0$.*

In particular, the elements $E, H, F \in M_{n+1}(\mathbb{F})$ are be represented by the matrices

$$[E]_{\mathcal{B}} = J_{n+1}(0) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

$$[H]_{\mathcal{B}} = \begin{bmatrix} -n & 0 & \cdots & 0 \\ 0 & -n+2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & n \end{bmatrix}, [F]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 \cdot n & \cdots & 0 & 0 \\ 0 & 0 & 2(n-1) & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & n \cdot 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

The theorem above says in particular that there is only one representation of $\mathfrak{sl}_2$ of dimension $n + 1$ (up to isomorphism). Furthermore any representation of $\mathfrak{sl}_2$ has a basis consisting of weight vectors. This justifies the following:

Definition 3.17. Let V be a representation of $\mathfrak{sl}_2$ and let $W_\lambda(V) = \{v \in V \mid Hv = \lambda v\}$ be the eigenspace corresponding to a weight (eigenvalue) λ for H. Then there is a decomposition $V = \bigoplus_\lambda W_\lambda(V)$ called the *weight space decomposition* of V.

Remark 3.18. If V is an irreducible representation for $\mathfrak{sl}_2$ and $\dim(V) = n+1$ then the weight spaces are the 1-dimensional spaces $W_{-n+2i}(V) = \mathbb{F}v_i$, with v_i as in Theorem 3.16.

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3.2.2 Weight space decompositions and SLPn

We now show that there is a close connection between artinian algebras satisfying SLPn and the representations of $\mathfrak{sl}_2$.

Remark 3.19. If A is a graded artinian $\mathbb{F}$-algebra and $L \in \mathfrak{m}$ then we can view A as a $\mathbb{F}[L]$-module since by the UMP of polynomial rings there exists a well defined ring homomorphism $\mathbb{F}[L] \to A$ which maps $L \mapsto L$. Since $\mathbb{F}[L]$ is a PID and A is a module over it, the structure theorem for modules over PIDs says that there is a module isomorphims

$$A \cong \mathbb{F}[L]/(p_1^{e_1}) \oplus \cdots \oplus F[L]/(p_k^{e_k})$$

where each p_i is a prime element of $\mathbb{F}[L]$ (no free part since A is finite dimensional). Since A is furthermore graded the elementary divisors $p_i^{e_i}$ must be homogeneous elements of $\mathbb{F}[L]$, thus $p_i =$ for all i. This implies that A decomposes as an internal direct sum as

$$A \cong S^{(1)} \oplus \cdots \oplus S^{(k)}, \text{ with } S^{(i)} \cong \mathbb{F}[L]/(L^{e_1}).$$

The cyclic $\mathbb{F}[L]$ modules $S^{(i)}$ are the *strands* of multiplication by L on A which were introduced before. This follows because the action of L on $S^{(i)}$ is given by a single Jordan block of size e_i.

Here is the connection between to $\mathfrak{sl}_2$:

Corollary 3.20. *The following are equivalent*

1. *S is a cyclic graded $\mathbb{F}[L]$ module i.e. $S \cong \mathbb{F}[L]/(L^d)$ (graded isomorphism, but not necessarily degree preserving)*
2. *$S = V_{d-1}$ is an irreducible representation of $\mathfrak{sl}_2$ with $Es = Ls$*

Proof. This follows because both the action of L on S as well as the action of E on V_d is given by a single Jordan block matrix. Once the basis of S has been fixed to be $1, L, L^2, \ldots, L^{d-1}$, the action of H and F can be simply defined to be the one given by the matrices displayed in Theorem 3.16. □

If we put the $\mathfrak{sl}_2$-module structures on the individual strands together we obtain

Theorem 3.21 (Theorem 3.32 in the book)**.** *Let A be a graded artinian algebra and let $L \in A_1$. The following are equivalent*

1. *L is a strong Lefschetz element on A in the narrow sense*
2. *A is an $\mathfrak{sl}_2$-representation with $E = \times L$ and the weight space decomposition of A coincides with the grading decomposition and in fact* weight$(v) = 2\deg(v) - c$*). This means that*

$$A = \bigoplus_{i=0}^{c} A_i = \bigoplus_{i=0}^{c} W_{2i-c}(A), \text{ where } A_i = W_{2i-c}(A).$$

3.3 Tensor products and the SLPn

From the above theorem, we can deduce how SLPn behaves when we take tensor products. We need

Lemma 3.22. *If A, A' are associative algebras which are representations of $\mathfrak{sl}_2$ (in particular A and A' have characteristic 0), then so is $A \otimes_{\mathbb{F}} A'$ with the action $g \cdot (v \otimes v') = (gv) \otimes v' + v \otimes (gv')$. If v, v' are weight vectors then $v \otimes v'$ is also a weight vector with* $\text{weight}(v \otimes v') = \text{weight}(v) + \text{weight}(v')$.

Proof. We show the statement about weights only: say $\text{weight}(v) = \lambda$ and $\text{weight}(v') = \lambda'$ so that $Hv = \lambda v, Hv' = \lambda v'$. Then

$$H(v \otimes v') = (Hv) \otimes v' + v \otimes (Hv') = \lambda v \otimes v' + v \otimes \lambda' v' = (\lambda + \lambda')v \otimes v'$$

shows that $v \otimes v'$ is a weight vector with weight $\lambda + \lambda'$. □

Theorem 3.23 (Theorem 3.34 in the book)**.** *Let $\mathbb{F}$ be a field of characteristic zero. If L is a strong Lefschetz element in the narrow sense on A and if L' is a strong Lefschetz element in the narrow sense on A' then $L \otimes 1 + 1 \otimes L'$ is a strong Lefschetz element in the narrow sense on $A \otimes_{\mathbb{F}} A'$.*

Proof. By Theorem 3.21 we have that $A_i = W_{2i-c}(A)$ and $A'_j = W_{2j-c'}(A')$, so

$$A = \bigoplus_{i=0}^{c} A_i = \bigoplus_{i=0}^{c} W_{2i-c}(A) \quad \text{and} \quad A' = \bigoplus_{j=0}^{c'} A'_j = \bigoplus_{j=0}^{c} W_{2j-c}(A')$$

imply

$$A \otimes_{\mathbb{F}} A' = \bigoplus_{i=0,j=0}^{c,c'} A_i \otimes_{\mathbb{F}} A'_j = \bigoplus_{i=0,j=0}^{c,c'} W_{2i-c}(A) \otimes_{\mathbb{F}} W_{2j-c'}(A').$$

From the fact that $\deg(v \otimes v') = \deg(v) + \deg(v')$ and the first equality above we deduce that

$$(A \otimes_{\mathbb{F}} A')_k = \bigoplus_{i=0}^{c} A_i \otimes_F A'_{k-i}.$$

Note that the maximum socle degree of $A \otimes_{\mathbb{F}} A'$ is $c + c'$. From the fact that $\text{weight}(v \otimes v') = \text{weight}(v) + \text{weight}(v')$ and the second equality above we deduce that

$$W_{2k-c-c'}(A \otimes_{\mathbb{F}} A') = \bigoplus_{i=0}^{c} W_{2i-c}(A) \otimes_{\mathbb{F}} W_{2(k-i)-c'} = \bigoplus_{i=0}^{c} A_i \otimes_{\mathbb{F}} A_{k-i}.$$

Comparing, we see that $(A \otimes_{\mathbb{F}} A')_k = W_{2k-c-c'}(A \otimes_{\mathbb{F}} A')$, where the weight spaces on $A \otimes_{\mathbb{F}} A'$ correspond to the action

$$E(v \otimes v') = Ev \otimes v' + v \otimes Ev' = Lv \otimes v' + v \otimes Lv' = (L \otimes 1 + 1 \otimes L)v \otimes v'.$$

Theorem 3.21 gives that $L \otimes 1 + 1 \otimes L$ is a strong Lefschetz element on $A \otimes_{\mathbb{F}} A'$. □

Corollary 3.24 (Stanley's Theorem - second proof)**.** *If $\mathbb{F}$ has characteristic 0, then the algebra $A = \mathbb{F}[x_1, \dots, x_n]/(x_1^{d_1}, \dots, x_n^{d_n}) = \mathbb{F}[x_1]/(x_1^{d_1}) \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathbb{F}[x_n]/(x_n^{d_n})$ satisfies SLP in the narrow sense.*

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A corollary of Theorem 3.23 is the following

Corollary 3.25. *If $\operatorname{char}(\mathbb{F}) = 0$ and A, A' are graded artinian $\mathbb{F}$-algebras which satisfy SLPn, then $A \otimes_{\mathbb{F}} A'$ also satisfies SLPn.*

Remark 3.26.

1. While the symmetric unimodality of Hilbert functions is preserved under taking tensor product, just unimodality is not. For example for

$$A = \mathbb{F}[x, y, z]/(x^2, xy, y^2, xz, yz, z^5)$$

with Hilbert function $1, 3, 1, 1, 1$ we have that the Hilbert function of $A \otimes_{\mathbb{F}} A$ is $1, 6, 11, 8, 9, 8, 3, 2, 1$.

2. While the SLPn is preserved under taking tensor product, the SLP (not in the narrow sense) is not preserved by tensor product. In the example above A has SLP but since its Hilbert function is not unimodal, $A \otimes_{\mathbb{F}} A$ cannot have the SLP.

The issue in part 2 of the remark is remedied by restricting to Gorenstein algebras, which have symmetric Hilbert function. Recall that for algebras with symmetric Hilbert function the SLP is equivalent to SLPn. Thus we have:

Corollary 3.27. *If $\operatorname{char}(\mathbb{F}) = 0$ and A, A' are graded artinian Gorenstein $\mathbb{F}$-algebras which satisfy SLP, then $A \otimes_{\mathbb{F}} A'$ also satisfies SLP.*

Chapter 4

Combinatorics of the Lefschetz properties

4.1 Posets and latices of all kinds

Definition 4.1. A *poset* (partially ordered set) is a set P endowed with a partial order $\leq$, i.e. a partial binary relation which is reflexive, anti-symmetric and transitive.

For a, b, elements of a partially ordered set P, if $a \leq b$ or $b \leq a$, then a and b are comparable. Otherwise they are incomparable. A partial order under which every pair of elements is comparable is called a total order or linear order. A poset (or subset of a poset) which is totally ordered is called a *chain*. The chain of cardinality $d+1$ is denoted $C(d)$. A subset of a poset in which no two distinct elements are comparable is called an *antichain*. For $a, b \in P$, we say that b *covers* a if $a < b$ and there are no elements properly between a and b.

A poset is called a *lattice* if each two-element subset $\{a, b\} \subseteq P$ has a *join* (i.e. least upper bound) and a *meet* (i.e. greatest lower bound), denoted by $a \vee b$ and $a \wedge b$ respectively.

We will now make an analogy between (finite) posets and graded Artinian algebras. Here is a summary:

Ring theory	**Poset theory**
$R = \mathbb{F}[x_1, \ldots, x_n]$	$P = \{x_1^{a_1} \cdots x_n^{a_n}\}, m \leq m'$ iff $m \mid m'$
graded artinian algebra $A = R/I$	finite poset $P(A) = \{$monomial basis for $A\}$ $m \leq m'$ iff $m \mid m'$
degree Hilbert function	rank Whitney numbers
multiplication by $L = x_i$ strands of multiplication by $L = x_i$	Hasse diagram disjoint chain decomposition for P
minimal set of generators for an ideal J	antichain
canonical module ω_A	dual poset $P^\vee$
$A = \mathbb{F}[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$ $A = \mathbb{F}[x_1, \ldots, x_n]/(x_1^{d_1}, \ldots, x_n^{d_n})$ A is Gorenstein, description given later	boolean lattice divisor lattice vector space lattice

Definition 4.2. Let A be an artinian algebra which can be presented as $A = \mathbb{F}[x_1, \ldots, x_n]/I$, with I a monomial ideal. We associate to A a lattice $P(A)$ with underlying set consisting of a basis of monomials for A. Thus the elements of $P(A)$ are exactly the monomials in $\mathbb{F}[x_1, \ldots, x_n]$ which are not in I. Notice that $P(A)$ is finite since A is Artinian.

The partial order for monomials $m, m' \in P(A)$ is given by $m \leq m'$ iff $m \mid m'$. Then $P(A)$ this is a lattice with $m \vee m' = \gcd(m, m')$ and $m \wedge m' = \operatorname{lcm}(m, m')$.

Definition 4.3. A poset P is said to be *ranked* if there exists a *rank function* $\rho : P \to \mathbb{N}$ such that $\rho(x) = 0$ if x is a minimal element and $\rho(y) > \rho(x)$ if $y > x$ and $\rho(y) = \rho(x)+1$ if y covers x. We call the sets

$$P_i = \{x \in P \mid \rho(x) = i\}.$$

the *rank sets* of P and their sizes $|P_i|$ are the rank sizes or *Whitney numbers* of P.

For a finite poset P with rank function ρ we have a partition $P = \bigcup_{i=0}^{c} P_i$.

Remark 4.4. The poset $P(A)$ of monomials in a graded Artinian algebra A of definition 4.2 is a ranked poset with rank function $\rho(x) = \deg(x)$. Thus $P_i = P \cap A_i$ and the decomposition $P = \bigcup_{i=0}^{c} P_i$ follows from $A = \bigoplus_{i=0}^{c} A_i$.

Definition 4.5. For a poset P, we define the *Hasse diagram* as

$$B(P) = \{(x, y) \mid y \text{ covers } x\},$$

which we usually depict by drawing an edge between x and y for every $(x, y) \in B(P)$.

Remark 4.6. If $P(A)$ is the poset of monomials in a graded Artinian algebra A and $m, m' \in P$ then m' covers m iff $m' = mx_i$ for some $x_i \in P_1$. Thus the covering relations are given by multiplication by a variable.

Remark 4.7. Recall that the strands of multiplication by L on A are sets $S^{(i)}$ such that

- $S^{(j)} \cong \mathbb{F}[L]/(L^{e_j})$ i.e. $S^{(j)} = \operatorname{Span}\{b_j, Lb_j, \ldots, L^{e_j-1}b_j\}$
- $A = \bigoplus_{j=1}^{r} S^{(j)}$

Suppose now that $L = x_i$ and the elements b_j are monomials for $1 \leq j \leq r$. Then

- the sets $C^{(j)} = \{b_j, Lb_j, \ldots, L^{e_j-1}b_j\}$ are chains in the poset $P(A)$
- the chains above are disjoint since the $S^{(j)}$ are disjoint subspaces of A and
- $P(A) = \bigcup_{j=1}^{r} C^{(j)}$ since each $C^{(j)}$ is a basis for $S^{(j)}$ and these chains form a partition for $P(A)$.

We call a partition of a poset into chains as above a *disjoint chain decomposition*.

Example 4.8 (Boolean lattice)**.** The boolean lattice, denoted $2^{[n]}$ is the set of all subsets of $[n] = \{1, 2, \dots, n\}$ ordered by containment. The rank function is $\rho(S) = |S|$. This is in bijection with monomials $x_1^{a_1} \cdots x_n^{a_n}$ where $a_i \in \{0, 1\}$ by setting $a_i = 1$ if $x_i \in S$ and $a_i = 0$ if $x_i \notin S$ for a given $S \subseteq [n]$. This shows that

$$2^{[n]} = P\left(\frac{\mathbb{F}[x_1, \dots, x_n]}{(x_1^2, \dots, x_n^2)}\right).$$

Example 4.9 (Divisor lattice)**.** Consider an integer $m \in N$. The divisor lattice $\mathcal{L}(m)$ is the set of all positive divisors of m ordered by divisibility. If $m = p_1^{d_1} \cdots p_n^{d_n}$ is the prime factorization of m with p_i distinct primes, then

$$\mathcal{L}(m) = \{p_1^{a_1} \cdots p_n^{a_n} \mid 0 \le a_i \le d_i\}$$

so the divisor lattice is the product

$$\mathcal{L}(m) = C(d_1) \times \cdots \times C(d_s),$$

where $(a_1, \dots, a_n) \le (b_1, \dots, b_n)$ iff $a_1 \le b_1, \dots, a_n \le b_n$ and rank function $\rho(a_1, \dots a_n) = a_1 + \cdots + a_n$. Note that there is a bijection

$$\mathcal{L}(m) \to P\left(\frac{\mathbb{F}[x_1, \dots, x_n]}{(x_1^{d_1}, \dots, x_s^{d_s})}\right) \qquad p_1^{a_1} \cdots p_n^{a_n} \mapsto x_1^{a_1} \cdots x_n^{a_n}$$

which is also a lattice isomorphism.

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Example 4.10 (Vector space lattice)**.** We define the vector space lattice $\mathcal{V}(n, q)$ to be the set of subspaces of the n-dimensional vector space $\mathbb{F}_q^n$ over the finite field $\mathbb{F}_q$ with q elements, ordered by the inclusion. The rank function is given by $\rho(V) = \dim V$.

4.2 Full matchings and the Sperner property

Definition 4.11. For a poset P, we denote the set of antichains of P by $\mathcal{A}(P)$ and define the *Dilworth number* of P as

$$d(P) = \max_{A \in \mathcal{A}(P)} |A|$$

.

Definition 4.12. A ranked poset P has the *Sperner property* if $d(P) = \max |P_i|$.

Example 4.13. The boolean lattice, the divisor lattice and the vector space lattice have the Sperner property. These assertions are each theorems in combinatorics proven by Sperner, Aigner and Kantor respectively.

We now look at how one might prove the Sperner property.

Theorem 4.14 (Dilworth)**.** *For a finite poset P , the Dilworth number $d(P)$ is equal to the minimum number of disjoint chains into which P can be decomposed.*

In order to find such disjoint chains we consider consecutive ranks of the poset. For a ranked poset P we define $B_i(P)$ to be the restriction of the Hasse diagram to ranks $i-1$ and i:

$$B_i(P) = \{(x, y) \mid x \in P_{i-1}, y \in P_i, y \text{ covers } x\}.$$

This set forms the edges of a bipartite graph, which we also call $B_i(P)$.

Definition 4.15. A matching in a bipartite graph G with $V(G) = X \times Y$ is a subset of disjoint edges $M \subseteq E(G)$ such that if $(x, y), (x', y') \in M$ then $x \neq x'$ and $y \neq y'$. The matching is *full* if $|M| = \min\{|X|, |Y|\}$.

We can deduce the Sperner property from having full matchings:

Proposition 4.16 (Theorem 1.31 in the book)**.** *Let P be a finite ranked poset with unimodal Whitney numbers. If each $B_i(P)$ has a full matching then P has the Sperner property.*

Proof. Take a full matching M_i for each $B_i(P)$ and consider the graph G with vertex set P and edge set $M = \bigcup M_i$. Then G is a union of k disjoint chains. This follows by observing that for any i each vertex of P_i belongs to at most two edges of G, one from B_i and one from B_{i+1}. Thus the connected components of G are chains and two chains never meet at any vertex since that would give two edges that cannot be a part of a matching. These chains partition P. Let $|P_j| = \max\{|P_i|\}$. Since there is exactly one chain passing through every element in P_j we have $k = |P_j|$. Moreover by Theorem 4.14 we have $k \geq d(P)$ and since P_j is an antichain we have $d(P) \geq |P_j|$. Then we must have $k = d(P) = |P_j|$. □

And it turns out that we can find such matchings provided that $P = P(A)$ and A has the WLP. In order to do this, define the *biadjacency matrix* $\mathrm{Biad}(B_i(P))$ of $B_i(P)$ to be the matrix of size $|P_{i-1}| \times |P_i|$ having entry 1 in row indexed by $a \in P_{i-1}$ and column indexed by $b \in P_i$ if and only if $(a, b) \in B_i(P)$.

Definition 4.17. The *permanent* of an $n \times n$ matrix M is

$$\mathrm{perm}(M) = \sum_{\sigma \in \Sigma_n} m_{1\sigma(1)} \cdots m_{n\sigma(n)}.$$

The following remarks are crucial to the relationship between the WLP and the Sperner property:

Remark 4.18.
- the permanent of a biadjacency matrix of a bipartite graph with two equal parts counts the number of full matchings of the graph. Indeed if we number the vertices of each part by the elements of $[n]$, the full matchings are of the form $(1, \sigma(1)), \ldots, (n, \sigma(n))$ for some $\sigma \in \Sigma_n$.

- if M is a matrix with nonnegative entries and $\det(M) \neq 0$ then $\text{perm}(M) \neq 0$. Indeed, since $\det(M) \neq 0$ it means that not all the terms in the permanent are zero, so $\text{perm}(T) > 0$.

Theorem 4.19. *If $A = \mathbb{F}[x_1, \ldots, x_n]/I$ is a graded Artinian algebra with I a monomial ideal and A has unimodal Hilbert function and satisfies the WLP then $P(A)$ has the Sperner property.*

Proof. As in one of the homework problems, if A has some weak Lefschetz element $L \in A_1$, after a linear change of variables we can obtain $L = x_1 + \cdots + x_n$. This uses the fact that I is a monomial ideal.

Note that the matrix of $\times L : A_i \to A_{i+1}$ is the same as $\text{Biad}(B_i(P))$. Since A has the WLP, it follows that, for each i, $\text{Biad}(B_i(P))$ has full rank. We will show that this implies the existence of a full matching in $B_i(P)$.

First, consider a nonzero submatrix M of $\text{Biad}(B_i(P))$ of size $\min\{|P_{i-1}|, |P_i|\}$ with $\det(M) \neq 0$ and restrict to the induced subgraph H of $B_i(P)$ on the vertices indexing the rows and columns of this minor. It suffices to find a full matching of this subgraph. By the above remarks, since $\det(M) \neq 0$ we deduce that $\text{perm}(M) > 0$ and therefore there is at least one full matching in $B_i(P)$.

Now proposition 4.16 and Dilworth's theorem 4.14 give the desired conclusion. □

Chapter 5

Gorenstein rings and Hessians

Lecture 13 – October 29, 2019

5.1 Duality for modules of finite length and Gorenstein rings.

The material in this section follows Eisenbud's Commutative Algebra book.

Recall the notion of a dual for an $\mathbb{F}$-vector space:

Definition 5.1. Let V be an $\mathbb{F}$-vector space. Its dual is

$$V^* = \operatorname{Hom}_{\mathbb{F}}(V, \mathbb{F}) = \{\varphi : V \to F \mid \varphi \text{ is } \mathbb{F}-\text{linear}\},$$

the vector space of linear functionals on V.

Remark 5.2. A few facts:

1. V^* is an $\mathbb{F}$ vector space with scalar multiplication given by $(a\varphi)(v) = a\varphi(v)$ for $a \in \mathbb{F}, \varphi \in V^*, v \in V$.

2. Suppose V is a finite dimensional vector space with basis $B = \{b_1, \ldots, b_n\}$. By the UMP of vector spaces, for each $1 \leq i \leq n$ there exists a dual functional $b_i^* \in V^*$ such that $b_i^*(b_j) = \delta_{ij}$. Then it can be shown that $B^* = \{b_1^*, \ldots, b_n^*\}$ is a basis for V^* and in particular $\dim_{\mathbb{F}}(V^*) = \dim_{\mathbb{F}}(V)$, so $V^* \cong V$.

3. In general, for an arbitrary collection of $\mathbb{F}$ vector spaces V_i we have

$$\left(\bigoplus_i V_i\right)^* \cong \prod_i V_i^*,$$

so in particular

$$\left(\bigoplus_{\text{finite}} V_i\right)^* \cong \bigoplus_{\text{finite}} V_i^*, \quad \text{but} \quad \left(\bigoplus_{\text{infinite}} V_i\right)^* \cong \prod_{\text{infinite}} V_i^* \not\cong \bigoplus_{\text{infinite}} V_i^*.$$

4. from 2. it follows that for finite dimensional vector spaces $(V^*)^* \cong V$. We will see below that this isomorphism (called Ψ) is more natural than the one between V and V^* because it does not depend on a choice of basis.

5. $(-)^*$ is an exact functor on the category of finite dimensional vector spaces, meaning that if

$$0 \to U \to V \to W \to 0$$

is an exact sequence of vector spaces then

$$0 \to W^* \to V^* \to U^* \to 0$$

is also an exact sequence. This is because any exact sequence of vector spaces is split and applying the Hom functor to a split exact sequence always yields an exact sequence.

We wish to extend this duality theory for finitely generated modules M over an artinian ring A. Notice that such a module is a finite dimensional $\mathbb{F}$-vector space, where $\mathbb{F}$ is the residue field of A. To make this more formal we define

Definition 5.3. A *dualizing functor* on the category of finitely generated A-modules is a contravariant A-linear functor D such that

1. $D^2 \cong \text{id}$ as functors, i.e. there is an A-module homomorphism Ψ such that for any A-modules M, N and A-module homomorphism $f : M \to N$ the diagram $\begin{array}{ccc} M & \xrightarrow{\Psi} & D^2(M) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^{**}} \\ N & \xrightarrow{\Psi} & D^2(N) \end{array}$ commutes and the rows are A-module isomorphisms.

2. D is exact in the sense that D takes exact sequences to exact sequences.

(actually one can omit the condition of exactness; see Exercise 21.2 in Eisenbud).

Example 5.4. If $A = \mathbb{F}$ is a field, then the functor $(-)^*$ is a dualizing functor by the properties listed above.

Definition 5.5. Let A be an artinian $\mathbb{F}$-algebra and let M be a finitely generated A-module. Define the dual of M to be $D(M) = \text{Hom}_{\mathbb{F}}(M, \mathbb{F})$.

Proposition 5.6. *$D(-)$ is a dualizing functor on the category of finitely generated A-modules.*

Proof. • Well definedness of $D(-)$ as a functor on A-modules:

The vector space $D(M)$ is an A-module by the action

$$(a\varphi)(m) = \varphi(am) \text{ for } \varphi \in D(M), a \in A, m \in M.$$

One can check that all the axioms for A-modules are satisfied. One can also check that given an A-module homomorphism $f : M \to N$, there is an induced A-module homomorphism

$$D(f) : D(N) = \mathrm{Hom}_{\mathbb{F}}(N, \mathbb{F}) \to D(M) = \mathrm{Hom}_{\mathbb{F}}(M, \mathbb{F})$$

given by

$$D(f)(\varphi) = \varphi \circ f.$$

This makes D a contravariant functor on the category of finitely generated Amodules.

- $D^2 \cong \mathrm{id}$

  Since M is finite-dimensional over $\mathbb{F}$, the evaluation map

  $$\Psi : M \to D(D(M))$$

  $$m \mapsto [\varphi \mapsto \varphi(m)]\,, \text{ for } \varphi \in \mathrm{Hom}_{\mathbb{F}}(M, \mathbb{F})$$

  is an isomorphism of vector spaces. Indeed it is true that $\dim_{\mathbb{F}}(D(M)) = \dim_{\mathbb{F}}(M)$, so also $\dim_{\mathbb{F}}(D^2(M)) = \dim_{\mathbb{F}}(D(M)) = \dim_{\mathbb{F}}(M)$. It is not hard to see that the evaluation map is injective because for every $m \neq 0$ there is a dual functional $m^* \in D(M)$ such that $m^*(m) = 1$. Since it is injective, Ψ it must be an isomorphism of vector spaces. In fact this is also an A-module homomorphism, hence it is in fact an A-module isomorphism.

- D is exact

  This is true because at the $\mathbb{F}$ vector space level $D(M) = M^*$ and so exactness of $D(-)$ follows from the exactenss of $(-)^*$.

□

Example 5.7. • Let $A = \mathbb{F}[x, y]/(x^2, xy^2, y^3)$.

Then $D(A) = \mathrm{Span}\{1^*, x^*, y^*, (xy)^*, (y^2)^*\}$ with A-module structure given by

	1^*	x^*	y^*	$(xy)^*$	$(y^2)^*$
1	1^*	x^*	y^*	$(xy)^*$	$(y^2)^*$
x	0	1^*	0	y^*	0
y	0	0	1^*	x^*	y^*
xy	0	0	0	1^*	0
y^2	0	0	0	0	1^*

We see that $D(A)$ has two generators as an A-module: $(xy)^*, (y^2)^*$, which are the dual elements to the socle elements of A. Since A only has one generator, 1, as an A-module we see that $A \ncong D(A)$ as A-modules, although they are isomorphic as vector spaces.

- Let $A = \mathbb{F}[x,y]/(x^2, y^3)$.

  Then $D(A) = \mathrm{Span}\{1^*, x^*, y^*, (xy)^*, (y^2)^*, (y^3)^*\}$ with A-module structure given by

	1^*	x^*	y^*	$(xy)^*$	$(y^2)^*$	$(xy^2)^*$
1	1^*	x^*	y^*	$(xy)^*$	$(y^2)^*$	$(xy^2)^*$
x	0	1^*	0	y^*	0	$(y^2)^*$
y	0	0	1^*	x^*	y^*	$(xy)^*$
xy	0	0	0	1^*	0	$(y)^*$
y^2	0	0	0	0	1^*	$(x)^*$
xy^2	0	0	0	0	0	1^*

  We see that $D(A)$ has one generator as an A-module: $(xy^2)^*$, and in fact as we shall see later this implies that $D(A) \cong A$ as A-modules.

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Let now A be a local or graded $\mathbb{F}$-algebra with (homogeneous) maximal ideal $\mathfrak{m}$. The main function served by the dualizing functor D is establishing a correspondence between the following

M		$D(M)$
$\mathrm{Ann}(M)$	$=$	$\mathrm{Ann}(D(M))$
socle of M		minimal generators of $D(M)$
$0 :_M \mathfrak{m}$	$\cong$	$D(M)/\mathfrak{m}D(M)$
M projective	$\Longleftrightarrow$	$D(M)$ injective

Lemma 5.8. *Suppose M is a finitely generated module over an artinian ring A. Then $0 :_M \mathfrak{m} \cong D(0 :_M \mathfrak{m}) \cong D(M)/\mathfrak{m}D(M)$ as A-modules.*

Proof. Since $D^2 \cong 1$ as functors, the map $\mathrm{Hom}_A(N, M) \to \mathrm{Hom}_A(D(M), D(N))$ given by $f \mapsto D(f)$ is an isomorphism. Thus there is an isomorphism $\mathrm{Hom}_A(\mathbb{F}, M) \cong \mathrm{Hom}_A(D(M), \mathbb{F})$ since $D(\mathbb{F}) = \mathbb{F}$. There are further isomorphisms

$$\begin{aligned} (0 :_M \mathfrak{m}) &\cong \mathrm{Hom}_A(A/\mathfrak{m}, M) = \mathrm{Hom}_A(\mathbb{F}, M) \\ \mathrm{Hom}_{\mathbb{F}}(D(M)/\mathfrak{m}D(M), \mathbb{F}) &\cong \mathrm{Hom}_A(D(M), \mathbb{F}) \end{aligned}$$

which give the isomorphism $(0 :_M \mathfrak{m}) \cong D\left(D(M)/\mathfrak{m}D(M)\right)$ equivalent to the claim. ☐

We can now finally define Gorenstein rings:

Definition 5.9. An artinian local or graded ring with (homogeneous) maximal ideal $\mathfrak{m}$ A is called *Gorenstein* if $A \cong D(A)$ as A-modules.

Example 5.10. Based on Example 5.7, the ring $\mathbb{F}[x,y]/(x^2, xy^2, y^3)$ is not Gorenstein, but $\mathbb{F}[x,y]/(x^2, y^3)$ is Gorenstein.

Theorem 5.11. *Let A be an artinian local or graded ring with residue field $\mathbb{F}$. The following are equivalent:*

1. *A is Gorenstein ($A \cong D(A)$).*
2. *$D(A)$ can be generated by one element as an A-module*
3. *$\dim_{\mathbb{F}}(0 :_A \mathfrak{m}) = 1$ i.e. the socle of A is 1-dimensional.*

Proof. (1) $\Rightarrow$ (2) is true by definition

(2) $\Rightarrow$ (1). Assuming (2), $D(A)$ is cyclic, i.e. of the form $D(A) \cong A/I$ for some ideal I of A. On the other hand comparing $\dim_{\mathbb{F}}(D(A)) = \dim_{\mathbb{F}}(A)$ and $\dim(D(A)) = \dim_F(A) - \dim_{\mathbb{F}}(I)$ yields that $\dim_{\mathbb{F}}(I) = 0$ and hence $I = 0$, so $D(A) \cong (A)$.

(3) $\Leftrightarrow$ (2) follows from Lemma 5.8. ☐

Looking back at Example 5.10 we see that the isomorphism $A \cong D(A)$ is not degree preserving. Indeed, since a generator of $D(A)$ is $(xy^2)^*$, this isomorphism must send $1 \mapsto (xy^2)^*$. However 1 is an element of degree 0 in A and $(xy^2)^*$ is an element of degree -3 in $D(A)$ because $(xy^2)^*$ is a homomorphism which decreases degrees by 3. So, the isomorphism $A \cong D(A)$ is not degree preserving. There is a way to fix this by shifting degrees.

Definition 5.12. For a graded ring A and an integer d, define $A(d)$ to be a the graded ring A with grading modified such that $A(d)_i = A_{d+i}$.

Now we can state the definition of Gorenstein rings that takes grading into account.

Definition 5.13. An artinian graded ring A is called *Gorenstein* if $A \cong D(A)(-c)$ as graded A-modules (degree preserving isomorphism), where c is the socle degree of A.

Proposition 5.14. *Let A be a graded artinian Gorenstein ring of socle degree c. Then*

1. *$A_i \cong A_{c-i}$ as vector spaces for any $0 \leq i \leq c$*
2. *the Hilbert function of A is symmetric.*

Proof. 1. $A \cong D(A)(-c)$ implies $A_i \cong D(A)(-c)_i = D(A)_{-c+i} = A^*_{c-i} \cong A_{c-i}$.

2. $A_i \cong A_{c-i}$ implies $H_A(i) = H_A(c-i)$ for any $0 \leq i \leq c$. ☐

Example 5.15. The Gorenstein ring $\mathbb{F}[x,y]/(x^2, xy^2, y^3)$ has symmetric Hilbert function $1, 2, 2, 1$.

On the other hand the ring $\mathbb{F}[x,y]/(x^2, xy^2, y^3)$ has symmetric Hilbert function $1, 2, 1$, but is not Gorenstein since its socle is the 2-dimensional vector space spanned by $\{x, y^2\}$.

It turns out that the dualizing functor $D(-)$ has a nice alternate expression:

Proposition 5.16. *If M is a finitely generated A-module then* $D(M) = \mathrm{Hom}_A(M, D(A))$.

Proof. Since $D^2 \cong 1$ as functors, the map $\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(D(N), D(M))$ given by $f \mapsto D(f)$ is an isomorphism. Thus there is an isomorphism

$$D(M) = \mathrm{Hom}_A(A, D(M)) \cong \mathrm{Hom}_A(D(D(M)), D(A)) \cong \mathrm{Hom}_A(M, D(A)).$$

□

Remark 5.17. If A is an arbitrary ring rather than an artinian local or graded one, then one obtains an analogous duality theory by setting $D(M) = \mathrm{Hom}_A(M, E)$, where E is an injective hull of the residue field $\mathbb{F}$.

Lecture 15 – November 5, 2019

Here are a few notions related to Gorenstein rings.

Definition 5.18. An artinian graded ring $A = \bigoplus_{i=0}^{c} A_i$ is *level* if the socle is concentrated in a single degree i.e. $0 :_A \mathfrak{m} = A_c$.

Remark 5.19. Note that in general $0 :_A \mathfrak{m} \supseteq A_c$. If A is graded artinian Gorenstein then $\dim_{\mathbb{F}}(0 :_A \mathfrak{m}) = 1$ by Theorem 5.11 and $\dim_{\mathbb{F}}(A_c) = \dim_{\mathbb{F}}(A_0) = 1$ by Poincaré duality, so we see that any graded artinian Gorenstein ring is level.

Definition 5.20. The *type* of an artinian ring A is the vector space dimension of its socle.

Remark 5.21. Lemma 5.8 says that the type of A is equal to the minimal number of generators for $D(A)$. Theorem 5.11 says that an artinian ring is Gorenstein if and only if it has type 1.

5.2 Graded duality and Macaulay's inverse systems

In the previous section, in order to define a meaningful duality we needed to work with finite length rings and modules. The same works for graded modules that are not necessarily finite dimensional $\mathbb{F}$-vector spaces but for which every graded component is a finite dimensional vector space.

Definition 5.22. For a graded $\mathbb{F}$-algebra A with $\dim_{\mathbb{F}}(A_i) < \infty$ for all i define *the graded homomorphism module* as follows:

$$D(A) = \mathrm{Hom}_{\mathbb{F}}^{\mathrm{gr}}(A, \mathbb{F}) = \bigoplus_{i \geq 0} \mathrm{Hom}_{\mathbb{F}}(A_i, \mathbb{F}).$$

As before, this is an A-module via $a\phi(x) = \phi(a \cdot x)$. If A is finite dimensional then $\mathrm{Hom}_{\mathbb{F}}^{\mathrm{gr}}(A, \mathbb{F}) \cong \mathrm{Hom}_{\mathbb{F}}(A, \mathbb{F})$ as A-modules so the graded hom dual agrees with the notion of dual from the previous section.

Example 5.23. Say $R = \mathbb{F}[x_1, \ldots, x_n]$ is the polynomial ring with the standard grading. We have

$$D(R) = \operatorname{Hom}_{\mathbb{F}}^{\mathrm{gr}}(R, \mathbb{F}) = \bigoplus_{i \geq 0} \operatorname{Hom}_{\mathbb{F}}(R_i, \mathbb{F}) = \bigoplus_{i \geq 0} \operatorname{Span}\{m^* \mid m \in R_i \text{ monomial }\}.$$

We'll make the convention from now on to write x_i^{-1} instead of the elements x_i^* in the dual basis of R_1^*. Then $D(R) = Q = \mathbb{F}[x_1^{-1}, \ldots, x_n^{-1}]$ viewed as an R-module under the action if $x_1^{a_1} \cdots x_n^{a_n} \in R$ and $x_1^{-b_1} \cdots x_n^{-b_n} \in Q$ are monomials, then

$$(x_1^{a_1} \cdots x_n^{a_n})(x_1^{-b_1} \cdots x_n^{-b_n}) = \begin{cases} x_1^{a_1 - b_1} \cdots x_n^{a_n - b_n} & \text{if } a_i - b_i \leq 0, \ \forall i \\ 0 & \text{otherwise.} \end{cases}$$

It turns out that the ring Q of Definition 5.22 is the injective hull of $\mathbb{F}$ and so it allows to define a dualizing functor on the category of finitely generated modules over the polynomial ring as in Remark 5.17.

Definition 5.24. If M is a graded module over the polynomial ring R, define

$$D(M) = \operatorname{Hom}_R(M, D(R)) = \operatorname{Hom}_R(M, Q).$$

More generally if M is a module over a complete local ring A with residue field $\mathbb{F}$ and E is the injective envelope of $\mathbb{F}$ one can define $D(M) = \operatorname{Hom}_A(M, E)$. In this generality one has the following:

Theorem 5.25 (Matlis duality)**.** *Let A be either a complete noetherian local ring or the polynomial ring. The functor D induces an anti-equivalence of categories between*

$$\{\textit{noetherian } A\textit{-modules}\} \leftrightarrow \{\textit{artinian } A\textit{-modules}\}$$

given both ways by sending $M \mapsto D(M)$.

Lemma 5.26. *Suppose $M = R/I$ for some homogeneous ideal I. Let's compute*

$$D(R/I) = \operatorname{Hom}_R(R/I, Q) \cong \operatorname{Ann}_Q(I) = (0 :_Q I) = \{g \in Q \mid fg = 0 \ \forall f \in I\}.$$

Example 5.27. Concretely, say

- $I = (x^2, y^3) \subseteq R = \mathbb{F}[x, y]$. Then $(0 :_Q I) = \operatorname{Span}\{x^{-1}y^{-2}, x^{-1}y^{-1}, y^{-2}, x^{-1}, y^{-1}, 1\} = R \cdot x^{-1}y^{-2}$ is the R-submodule of Q generated by $x^{-1}y^{-2}$.
- $I = (x^2, xy^2, y^3) \subseteq R = \mathbb{F}[x, y]$. Then $(0 :_Q I) = \operatorname{Span}\{x^{-1}y^{-1}, y^{-2}, x^{-1}, y^{-1}, 1\} = R \cdot x^{-1}y^{-1} + R \cdot y^{-2}$ is an R-submodule of Q with two generators.

Notice in both cases that the number of generators of $D(R/I)$ is equal to the dimension of the socle of R/I as proven in Lemma 5.8 (which applies since R/I is Artinian in both cases).

Next, take

- $I = (x) \subseteq R = \mathbb{F}[x, y]$. Then $(0 :_Q I) = \operatorname{Span}\{y^{-i} \mid i \geq 0\}$.
- $I = (x^d) \subseteq R = \mathbb{F}[x, y]$. Then $(0 :_Q I) = \operatorname{Span}\{x^{-i}y^{-j} \mid 0 \leq i \leq d-1, j \geq 0\} = Q_0 \oplus Q_1 \oplus Q_2 \oplus \cdots \oplus Q_{d-1} \oplus yQ_{d-1} \oplus y^2Q_{d-1} \oplus \cdots \oplus y^kQ_{d-1} \oplus \cdots$.

Both of the above $(0 :_Q I)$ are non-finitely generated R-module. We shall see below that this corresponds to R/I not being artinian, i.e $\dim_{\mathbb{F}}(R/I) = \infty$.

Matlis duality for the polynomial ring can be stated in concrete terms as follows:

Theorem 5.28 (Macaulay inverse system duality)**.** *With notation as above, there are bijective correspondence between*

$$\{\text{graded } R-\text{modules } M \subseteq Q\} \leftrightarrow \{R/I \mid I \subseteq R \text{ homogeneous ideal}\}$$

$$M \mapsto D(M) \cong R/(0 :_R M)$$

$$(0 :_Q I) = \{m \in Q \mid Im = 0\} = D(R/I) \leftarrow\!\shortmid R/I$$

Furthermore, we have the additional correspondences

$$\begin{array}{llcc} (a) & M \text{ finitely generated} & \iff & R/(0 :_R M) \text{ artinian} \\ (b) & M = R \cdot F \text{ cyclic} & \iff & R/(0 :_R F) \text{ artinian Gorenstein} \\ & \deg(F) & = & \text{socle degree of } R/(0 :_R F) \end{array}$$

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Remark 5.29. Don't let the notation deceive you! If I is an ideal of R, it does not mean that $(0 :_Q I)$ is an ideal (or Q-submodule) of Q. It is just an R-module which happens to be a subset of Q.

Proof of Theorem 5.28. Note that the description of the correspondence above fits with Matlis duality in that it is given both ways by taking the dual of an R-module in the sense of Definition 5.24. We have shown in Lemma 5.26 that $D(R/I) = (0 :_Q I)$.

It remains to show that $D(M) \cong R/(0 : RM)$ for $M \subseteq Q$. Using the fact that $D(-)$ is an exact functor, the inclusion $M \hookrightarrow Q$ dualizes to a surjection $R = D(Q) \twoheadrightarrow D(M)$. Then the kernel of this homomorphism is $\operatorname{Ann}_R(D(M)) = \operatorname{Ann}_R(M) = (0 :_R D(M))$ and so $D(M) \cong R/(0 :_R M)$.

Finally, this correspondence is a bijection because $D^2 = \mathrm{id}$.

(a) If M is finitely generated then a high enough power of $\mathfrak{m}$ is contained in $(0 :_R M)$ by degree reasons, hence $R/(0 :_R M)$ is Artinian. If R/I is artinian we have $M = D(R/I)$ is generated by the type of R/I many generators by Lemma 5.8, which is a finite number of generators.

(b) We know that R/I is artinian Gorenstein iff $D(R/I) = R \cdot F$ is a cyclic R-module by Theorem 5.11. Now we can make the isomorphism $R/I \cong D(R/I)(-c)$, where c is the socle degree of A, explicit by sending $1 \mapsto F$ and hence $r \mapsto rF$ for any $r \in R$. Since this isomorphism preserves degrees, $F \in D(R/I)(-c)_0$, so $F \in D(R/I)_{-c}$ i.e. $\deg(F) = c$ in the variables of Q. Moreover $F \in D(R/I)_c = A_c^*$ is the function $A_c \to \mathbb{F}$, $b \mapsto bF$. ∎

The value of the above theorem often lies in producing examples of artinian Gorenstein rings.

Definition 5.30. The polynomial $F \in Q$ is called a *dual socle generator* for $R/(0 :_R F)$.

Example 5.31. The artinian Gorenstein algebra with dual socle generator

$$F = x^{-2} + y^{-2} + z^{-2}$$

is

$$\mathbb{F}[x, y, z]/(0 :_{\mathbb{F}} [x, y, z]F) = \mathbb{F}[x, y, z]/(x^2 - y^2, y^2 - z^2, z^2 - x^2, xy, xz, yz).$$

The artinian Gorenstein algebra with dual socle generator

$$F = x_1^{-d_1} \cdots x_n^{-d_n}$$

is the monomial complete intersection

$$\mathbb{F}[x, y, z]/(0 :_{\mathbb{F}} [x, y, z]F) = \mathbb{F}[x, y, z]/(x_1^{d_1+1}, \ldots, x_n^{d_n+1}).$$

5.3 SLP for Gorenstein rings via Hessians

For this section let $R = \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial ring and Q its graded dual. We will further assume that $\mathrm{char}(\mathbb{F}) = 0$.

Fact 5.32. *Q is isomorphic to $\mathbb{F}[X_1, \ldots, X_n]$ the R-module with R-action $x_i F = \frac{\partial F}{\partial X_i}$. We will use this description for Q in this section.*

Lemma 5.33. *Let $F \in Q_c$ and let $L = a_1x_1 + \cdots + a_nx_n \in R_1$. Then*

$$L^c F = c! \cdot F(a_1, \ldots, a_n).$$

Proof.

$$L^c F = \sum_{i_1+\cdots+i_n=c} \frac{c!}{i_1! \cdots i_n!} a_1^{i_1} \cdots a_n^{i_n} x_1^{a_1} \cdots x_n^{i_n} F = c! \cdot F(a_1, \ldots, a_n).$$

□

Definition 5.34. Let $F \in Q$ be a homogeneous polynomial and let $B = \{b_1, \ldots, b_s\} \subseteq R_d$ be a finite set of homogeneous polynomials of degree $d \geq 0$. We call the polynomial $\mathrm{Hess}_B^d(F) = \det [b_i b_j F]_{1 \leq i,j \leq s}$ the d-th *Hessian* of F with respect to B.

Remark 5.35. If $B = \{x_1, \ldots x_n\}$ then $\mathrm{Hess}_B^1(F) = \det [x_i x_j F]_{1 \leq i,j \leq n} = \det \left[\frac{\partial F}{\partial X_i \partial X_j}\right]_{1 \leq i,j \leq n}$ is the classical Hessian of F.

Hessians are useful in establishing the SLP for artinian Gorenstein rings.

Theorem 5.36. *Assume $\mathbb{F}$ is an infinite field. Let A be a graded artinian Gorenstein ring with dual socle generator $F \in Q_c$. Then A has the SLP if and only if*

$$\mathrm{Hess}^i_{B_i}(F) \neq 0 \text{ for } 0 \leq i \leq \lfloor \frac{c}{2} \rfloor$$

where B_i is some (any) basis of A_i.

Example 5.37. Say $F = X^2 + Y^2 + Z^2$. Then with respect to the standard monomial basis for each R_i

$$\begin{aligned} \mathrm{Hess}^0(F) &= F \\ \mathrm{Hess}^1(F) &= \det \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 1 \\ \mathrm{Hess}^i(F) &= 0 \text{ for } i \geq 2. \end{aligned}$$

Example 5.38. Let $G = XYW^3 + X^3ZW + Y^3Z^2$. Then $A = R/(0 :_R G)$ has Hilbert function $1, 4, 10, 10, 4, 1$ and a basis for A_1 is $B_1 = \{x, y, z, w\}$ whereas a basis for A_2 is $B_2 = \{x^2, xy, xz, xw, y^2, yz, yw, z^2, zw, w^2\}$. Furthermore

$$\begin{aligned} \mathrm{Hess}^0(G) &= G \\ \mathrm{Hess}^1_{B_1}(G) &= \det \begin{pmatrix} 6XZW & W^3 & 3X^2W & 3X^2Z + 3YW^2 \\ W^3 & 6YZ^2 & 6Y^2Z & 3XW^2 \\ 3X^2W & 6Y^2Z & 2Y^3 & X^3 \\ 3X^2Z + 3YW^2 & 3XW^2 & X^3 & 6Z^2W \end{pmatrix} \neq 0 \\ \mathrm{Hess}^2_{B_2}(G) &= 0. \end{aligned}$$

We conclude that the map $L : A_2 \to A_3$ fails to have maximum rank for all $L \in A_1$. However the map $L^3 : A_1 \to A_4$ does have maximum rank.

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Proof. From the hypotheses we have that $A = R/(0 :_R F)$ has socle degree $c = \deg(F)$.

Since A is Gorenstein, A has symmetric Hilbert function, so A has SLP if and only if A has SLP in the narrow sense, i.e. there exists $L \in A_1$ such that for any $0 \leq i \leq \lfloor \frac{c}{2} \rfloor$ the multiplication maps $L^{c-2i} : A_i \to A_{c-i}$ are vector space isomorphisms. Say $L = a_1x_1 + \cdots + a_nx_n$.

Recall that the isomorphism $A \cong D(A)(-c) = (RF)(-c), a \mapsto aF$ induces vector space isomorphisms $A_{c-i} \cong A_i^*$ also defined by $a \mapsto [b \mapsto b(aF)]$. The composite map

$$T_i : A_i \xrightarrow{L^{c-2i}} A_{c-i} \xrightarrow{F} A_i^*$$

is an isomorphism if and only if multiplication by L^{c-2i} is an isomorphism. Let B_i be any basis for A_i and let B_i^* be its dual, which is a basis for A_i^*. The matrix $[t_{jk}^{(i)}]$for T_i with respect to these bases is defined as follows

$$T_i(b_j) = \sum_{k=1}^{s} t_{jk}^{(i)} b_k^*,$$

hence $t_{jk}^{(i)} = T_i(b_j)(b_k) = F(b_j L^{c-2i})(b_k) = (c-2i)!(b_j b_k F)(a_1, \dots, a_n)$, thus T_i is an isomorphism for some $L \in R_1$ if and only if

$$\mathrm{Hess}_{B_i}^i(a_1, \dots, a_n) = \det\left[b_i b_j F(a_1, \dots, a_n)\right]_{1 \le i,j \le s} \neq 0.$$

Overall the SLP holds if and only if for $0 \le i \le \lfloor \frac{c}{2} \rfloor$ the hessian polynomial $\mathrm{Hess}_{B_i}^i$ does not vanish identically. □

Corollary 5.39. *Let $F \in \mathbb{F}[X_1, \dots, X_n], G \in \mathbb{F}[Y_1, \dots, Y_n]$ be homogeneous polynomials of the same degree. Then $A = \mathbb{F}[x_1, \dots, x_n]/\operatorname{Ann}(F)$ and $B = \mathbb{F}[y_1, \dots, y_n]/\operatorname{Ann}(G)$ have SLP if and only if*

$$C = \mathbb{F}[x_1, \dots, x_n, y_1, \dots, y_n]/\operatorname{Ann}(F+G) \text{ satisfies SLP.}$$

Proof. It turns out that a basis β of C_i is given by the union of a basis β' of A_i and a basis β'' of B_i and hence the hessians of $F+G$ look like

$$\begin{aligned}\mathrm{Hess}^i(F+G) &= \det\left[b_i b_j (F+G)\right]_{b_i, b_j \in \beta} = \det\begin{bmatrix} b_i' b_j'(F) & 0 \\ 0 & b_i'' b_j''(F)\end{bmatrix}_{b_i', b_j' \in \beta', b_i'', b_j'' \in \beta''} \\ &= \mathrm{Hess}^i(F)\,\mathrm{Hess}^i(G).\end{aligned}$$

Now we see that $\mathrm{Hess}^i(F+G) \neq 0$ if and only if $\mathrm{Hess}^i(F) \neq 0$ and $\mathrm{Hess}^i(G) \neq 0$ which gives the desired conclusion. □

Chapter 6

Open questions

Lecture 18 – Nov 21, 2019

I list some of the most important and currently open questions on Lefschetz properties. The Jordan type, non-Lefschetz locus and Sperner property are currently much less explored, so these are both open in pretty much any context you can think of.

6.1 Gorenstein rings

We have seen in Example 5.38 of section 5 that there is an artinian Gorenstein ring of embedding dimension 4 that does not have the WLP. No such examples are known in embedding dimension three in characteristic zero, although examples of artinian Gorenstein rings of embedding dimension 3 and postitive characteristic failing the WLP are known. Therefore one can ask:

Question 6.1. *Do all artinian Gorenstein algebras of embedding dimension three having characteristic zero satisfy the WLP or SLP?*

In the paper [1] a reduction of this problem is given to a very specific type of Gorenstein algebra. More precisely, it is shown that in order to prove that the WLP holds for all artinian Gorenstein algebras of codimension 3, it is enough to prove that it holds for all *compressed* artinian Gorenstein algebras of odd socle degree $c = 2t - 1$; that is, for artinian Gorenstein algebras having Hilbert function

$$1, 3, 6, \dots, \binom{t}{2}, \binom{t+1}{2}, \binom{t+1}{2}, \binom{t}{2}, \dots, 6, 3, 1$$

It has also been shown in the same paper that compressed algebras satisfy WLP for socle degrees up to 5.

6.2 Complete intersections

Definition 6.2. An artinian ring $A = R/I$ with $R = \mathbb{F}[x_1, \dots, x_n]$ is called a *complete intersection* if I can be generated by n polynomials.

Fact 6.3. *Every complete intersection is Gorenstein.*

Stanley's Theorem 2.42 shows that any monomial complete intersection (i.e any complete intersection $A = R/I$ where I can be generated by monomials) having characteristic zero satisfies the SLP. This can be used to deduce that "most" complete intersections of characteristic zero also satisfy the SLP. However the central open question in the theory of the algebraic Lefschetz properties is whether *all* complete intersections of characteristic zero have SLP.

In the paper [3] it was proven that every complete intersections of characteristic zero and embedding dimension three has the weak Lefschetz property and it was conjectured that

Conjecture 6.4 ([3])**.** *All artinian complete intersections having characteristic zero satisfy the SLP.*

In fact the above conjecture is equivalent (see Proposition 3.44 in the textbook) to the apparently weaker

Conjecture 6.5. *All artinian complete intersections having characteristic zero satisfy the WLP.*

6.3 Modules over exterior algebra

Recall that the exterior algebra $E = \bigwedge_{\mathbb{F}}[e_1, \ldots, e_n]$ on variables $x_1, \ldots, x_n$ is the non-commutative algebra generated by these variables subject to the relations $x_i x_j = -x_j x_i$ for $i \neq j$ and $x_i^2 = 0$. We consider E as a graded algebra with $\deg(x_i) = 1$ and we single out the even subalgebra

$$E_{\text{even}} = \bigoplus_{n \geq 0} E_{2n}.$$

Note that E_{even} is a commutative ring because elements of even degree commute in E and furthermore that E is a module over E_{even}. It follows that any E-module M is also an E_{even}-module. One may want to rescale degrees in E_{even} by halving degrees so that elements of E_{2n} have degree n in E_{even}. Recently, J. Watanabe asked the following question:

Question 6.6. *Which E-modules M have the WLP or the SLP as E_{even}-modules?*

In effect this question asks whether there is an element of E_2 whose powers induce multiplication maps of maximal rank on M.

The case $M = E$ is best understood: recall from Example 1.6 that for $\mathbb{F} = \mathbb{C}$, E is the cohomology ring of the complex torus. So the Hard Lefschetz Theorem assures there exists an element $\omega \in E_2$ such that the multiplication maps $E_i \xrightarrow{\omega^{n-2i}} E_{n-i}$ are isomorphisms for $0 \leq i \leq n$. Since E has symmetric Hilbert function it follows that E has SLP. A different proof of this fact, which also holds in positive (but large enough) prime characteristic can be found in the appendix of the paper [2].

Bibliography

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